CO255 **Assignment 1** **Due: September 21**

Submit your assignment at the start of class. If a solution is not essentially correct you will get no credit. You may discuss assignment solutions with another student as long as neither of you has yet written a solution; taking written notes during the discussion is considered cheating.

Problem 1: Let $a_1, \ldots, a_m \in \mathbf{R}^n$ and $b_1, \ldots, b_m \in \mathbf{R}$. For $x \in \mathbf{R}^n$, we define $\epsilon_i = |a_i^t x - b_i|$, for each $i \in \{1, \ldots, m\}$, and let $\epsilon = \max(\epsilon_1, \ldots, \epsilon_m)$. We would like to choose $x \in \mathbf{R}^n$ minimizing ϵ. Formulate this problem as a linear program.

Problem 2: A set $S \subseteq \mathbf{R}^n$ is *convex* if for each $x, y \in S$ and $0 \leq \lambda \leq 1$, we have $\lambda x + (1 - \lambda) y \in S$.

(a) Prove that, if $S_1 \subseteq \mathbf{R}^n$ and $S_2 \subseteq \mathbf{R}^n$ are both convex, then $S_1 \cap S_2$ is convex.

(b) Let $a \in \mathbf{R}^n$ and $b \in \mathbf{R}$. Prove that $\{x \in \mathbf{R}^n : a^t x \leq b\}$ is convex. (Note that: by (a) and (b), the feasible region of any linear program is convex.)

Problem 3: A point $x \in \mathbf{R}^n$ is an *extreme point* of $S \subseteq \mathbf{R}^n$ if $x \in S$ and there do not exist two distinct points $a, b \in S$ and a real number $0 < \lambda < 1$ such that $x = \lambda a + (1 - \lambda) b$. A *line* in $\mathbf{R}^n$ is a set $\{x + \lambda d : \lambda \in \mathbf{R}\}$ where $x, d \in \mathbf{R}^n$ and $d \neq 0$.

Let $S = \{x \in \mathbf{R}^n : Ax \leq b\}$ where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$.

(a) Let $x^* \in S$ and let $A'x \leq b'$ be the subsystem of $Ax \leq b$ that x^* satisfies with equality. Prove that x^* is an extreme point of S if and only if $\text{rank}(A') = n$.

(b) Prove that, if S is nonempty, then the following are equivalent

(i) S has no extreme point.

(ii) S contains a line.

(iii) the columns of A are linearly dependent.

(c) Let $c \in \mathbf{R}^n$ and suppose that $\min(c^t x : x \in S)$ has an optimal solution. Prove that, if the first column of of A is spanned by the others, then there exists an optimal solution x^* to $\min(c^t x : x \in S)$ with $x_1^* = 0$.

Problem 4: Prove or disprove the following assertions:

(a) If $S \subseteq \mathbf{R}^n$ is not convex, then there exists $x, y \in S$ such that $\frac{1}{2}(x + y) \notin S$.

(b) If $S \subseteq \mathbf{R}^n$ is convex and $x \in S$ is not an extreme point, then there exist distinct $a, b \in S$ such that $x = \frac{1}{2}(a + b)$.

(c) If $S \subseteq \mathbf{R}^n$ is closed convex set, $f : S \to \mathbf{R}$ is a linear function, and there exists $l \in \mathbf{R}$ such that $f(x) \geq l$ for all $x \in S$, then there exists a minimizer $x \in S$ of f. (We say that S is *closed* if each convergent sequence of points in S has its limit in S.)

(d) [Bonus Problem] If $S \subseteq \mathbf{R}^n$ is an unbounded convex set, then there exists $x \in S$ and $d \in \mathbf{R}^n$ such that $\{x + \lambda d : \lambda \geq 0\} \subseteq S$. ($S$ is *bounded* if there exists $d \in \mathbf{R}$ such that S is contained in the cube $[-d, d]^n$.)

(e) [Bonus Problem] If $S \subseteq \mathbf{R}^n$ is a convex set, and $c \in \mathbf{R}^n$ such that $c^t x$ is not bounded above on S, then there exists $x \in S$ and $d \in \mathbf{R}^n$ such that $c^t d > 0$ and $\{x + \lambda d : \lambda \geq 0\} \subseteq S$.