CO255 **Assignment 2** **Due: October 5**

Submit your assignment at the start of class. If a solution is not essentially correct you will get no credit. You may discuss assignment solutions with another student as long as neither of you has yet written a solution; taking written notes during the discussion is considered cheating.

Problem 1: Consider the following system of inequalities:

$$\begin{array}{rcrcr} 3x_1 & - & x_2 & \leq & 3 \\ x_1 & + & x_2 & \leq & 2 \\ 2x_1 & + & 2x_2 & \geq & -1 \\ 2x_1 & & & \geq & 3 \end{array}$$

(a) Using Fourier-Motzkin Elimination, eliminate the variable x_2.

(b) Show that the original system is infeasible by giving an appropriate linear combination of the original constraints.

Problem 2:

(a) Let $P = \{x \in \mathbf{R}^n : Ax = b, x \geq 0\}$ where $A \in \mathbf{R}^{m\times n}$, with $\text{rank}(A) = m$, and $b \in \mathbf{R}^m$. We call $B \subseteq \{1, \dots, n\}$ a *basis* if $|B| = m$ and $\text{rank}(A_B) = m$. If B is a basis, the associated *basic solution* is the unique $x \in \mathbf{R}^n$ such that $Ax = b$ and $x_i = 0$ for all $i \notin B$. A *basic feasible solution* is a basic solution that is non-negative. Prove that $x \in \mathbf{R}^n$ is an extreme point of P if and only if it is a basic feasible solution. (Hint: You may use results from assignment 1.)

(b) Let Z be a finite set of points in $\mathbf{R}^m$ and let $x \in \text{conv}(Z)$. Prove that there is a set $Z' \subseteq Z$ such that $x \in \text{conv}(Z')$ and $|Z'| \leq m + 1$. (Hint: Use part (a).)

Problem 3: Let $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$. Prove that, if the system $Ax \leq b$ is infeasible, then there is an infeasible subsystem with at most $n + 1$ inequalities. (Hint: Use the Farkas Lemma and consider the matrix $[A, b]$.)

Problem 4: Let $z, A_1, \dots, A_n \in \mathbf{R}^m$. Prove that, if $z \notin \text{conv}(A_1, \dots, A_n)$ then there exist $\alpha \in \mathbf{R}^m$ and $\beta \in \mathbf{R}$ such that $\alpha^t x > \beta$ and $\alpha^t A_i \leq \beta$ for each $i \in \{1, \dots, n\}$. (Hint: Apply the Farkas Lemma.)

Problem 5:[Bonus Problem] Let $P \subseteq \mathbf{R}^n$ be a polyhedron. Prove that a point $x \in \mathbf{R}^n$ is an extreme point of P if and only if there is a hyperplane H such that $H \cap P = \{x\}$.