CO255 **Assignment 3** **Due: October 19**

Submit your assignment at the start of class. If a solution is not essentially correct you will get no credit. You may discuss assignment solutions with another student as long as neither of you has yet written a solution; taking written notes during the discussion is considered cheating.

Problem 1: Let $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, and $c \in \mathbf{R}^n$.

(a) Consider the following systems:

$$(1)\quad \left\{ \begin{array}{rrcl} -c^T x & +b^T y & \leq & 0 \\ Ax & & \leq & b \\ & -A^T y & = & -c \\ & y & \geq & 0 \end{array} \right. \qquad (2)\quad \left\{ \begin{array}{rrcl} -c^T x & +b^T y & < & 0 \\ Ax & & \leq & zb \\ & A^T y & = & zc \\ & y & \geq & 0 \\ & z & \geq & 0 \end{array} \right.$$

Prove that, if (1) is infeasible, then (2) is feasible. (Hint: Use the Farkas Lemma.)

(b) Consider the following LP and its dual:

$$(P) \quad \max(c^T x \,:\, Ax \leq b)$$

$$(D) \quad \min(b^T y \,:\, A^T y = c,\, y \geq 0)$$

Prove that, if (D) has an optimal solution $\hat{y}$, then (P) has an optimal solution $\hat{x}$, and $b^T\hat{y} = c^T\hat{x}$. (Do not reduce this to other versions of the Strong Duality Theorem, prove it following the proof of the Strong Duality Theorem.)

Problem 2: Consider the LP:

$$(P) \quad \max(x_1 + x_2 \,:\, x_1 - x_2 \leq -1,\, -x_1 + x_2 \leq -1, x_1, x_2 \geq 0).$$

(a) Write the dual (D) of (P).

(b) Show that (P) and (D) are both infeasible.

Problem 3: Let $A \in \mathbf{R}^{m\times n}$. Prove that, there exist $c \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$ so that $\max(c^T x \,:\, Ax \leq b)$ and its dual are both infeasible if and only if there exist a nonzero vector $d \in \mathbf{R}^m$ satisfying $(A^T d = 0,\, d \geq 0)$ and a nonzero vector $z \in \mathbf{R}^n$ satisfying $Az \leq 0$.

Problem 4: Prove that, if $K = \{x \in \mathbf{R}^n \,:\, Ax \leq 0\}$, then there exist points $z_1, \ldots, z_s \in K$ such that $K = \text{cone}(z_1, \ldots, z_s)$. (Hint: Consider the proof of Corollary 5 in the class notes.)

Problem 5: Let $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$ with $b \geq 0$. Consider the system:

$$(P) \quad \min \left(\sum_{i=1}^{m} s_i \; : \; Ax + s = b, \; x \geq 0, \; s \geq 0 \right).$$

(a) Show that (P) has an optimal solution. (Hint: You may use the Fundamental Theorem.)

(b) Show that the optimal value of (P) is 0 if and only if the system $(Ax = b, \; x \geq 0)$ is feasible.

Problem 6: [Bonus Problem] Let $A \in \mathbf{R}^{m\times n}$.

(a) Show that, if $\bar{p} \in \mathbf{R}^n$ satisfies $(p_1 + \cdots + p_n = 1, \; p \geq 0)$ and $\bar{q} \in \mathbf{R}^n$ satisfies $(q_1 + \cdots + q_m = 1, \; q \geq 0)$, then

$$\min_{i\in\{1,\ldots,m\}} \sum_{j=1}^{n} a_{ij}\bar{p}_j \quad \leq \quad \max_{j\in\{1,\ldots,n\}} \sum_{i=1}^{m} a_{ij}\bar{q}_i.$$

(b) Show that there exist $\bar{p}$ and $\bar{q}$ attaining equality. (Hint: Remember Rose and Colin?)