CO255 **Assignment 4** **Due: November 2**

Submit your assignment at the start of class. If a solution is not essentially correct you will get no credit. You may discuss assignment solutions with another student as long as neither of you has yet written a solution; taking written notes during the discussion is considered cheating.

Problem 1: In the following two parts, apply the simplex method starting from the basis $B = \{1, 2, 3\}$. If the problem has an optimal, give the optimal solution for the primal and the dual; if the problem is unbounded, show a feasible half-line with unbounded objective value.

(a)

$$(\mathrm{P})\left\{\begin{array}{lrrrrrcl} \text{max} & & & & 2x_4 & +x_5 & & \\ \text{subject to} & x_1 & & & -x_4 & +x_5 & = & 1 \\ & & x_2 & & +x_4 & +3x_5 & = & 3 \\ & & & x_3 & +x_4 & +x_5 & = & 2 \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0. \end{array}\right.$$

(b)

$$(\mathrm{P})\left\{\begin{array}{lrrrrrcl} \text{max} & & & & -x_4 & +2x_5 & & \\ \text{subject to} & x_1 & & & +x_4 & -2x_5 & = & 1 \\ & & x_2 & & +3x_4 & & = & 4 \\ & & & x_3 & +x_4 & -x_5 & = & 2 \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0. \end{array}\right.$$

Problem 2: Consider the following linear program.

$$(\mathrm{P})\left\{\begin{array}{lrrrrrcl} \text{max} & x_1 & & +x_3 & +2x_4 & +2x_5 & & \\ \text{subject to} & 2x_1 & -x_2 & +x_3 & & -3x_5 & = & 2 \\ & -3x_1 & +2x_2 & & & +7x_5 & = & 0 \\ & 2x_1 & -x_2 & & +x_4 & -4x_5 & = & 2 \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0. \end{array}\right.$$

(a) Consider applying the Simplex Method starting from the basis $B = \{1, 2, 3\}$. Give all pairs of valid entering and leaving variables.

(b) Apply the Perturbation Method, changing the right hand sides of the equations to $(2 + \epsilon, 0 + \epsilon^2, 2 + \epsilon^3)$. Now give all pairs of valid entering and leaving variables.

Problem 3: Let $\mathbf{R}_+$ denote $\{x \in \mathbf{R} : x \geq 0\}$. For each part, determine whether the statement is true or false; justify your answer.

(a) If $P \subseteq \mathbf{R}^n_+$ is an unbounded integral polyhedron, then P contains an unbounded number of integral points.

(b) If $P \subseteq \mathbf{R}^n_+$ is an integral polyhedron that contains no integer points, then P is empty.

(c) If $P \subseteq \mathbf{R}^n$ is an integral polyhedron that contains no integer points, then P is empty.

Problem 4: Show that the convex hull of the integer solutions to $\{x \in \mathbf{R}^2 : x_1 - \sqrt{2}x_2 \geq 0,\ x_1, x_2 \geq 0\}$ is not a polyhedron. (You may use without proof that $\sqrt{2}$ is irrational.)

Problem 5:

(a) Let $\bar{x}$ be an extreme point of a polyhedron $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ where $A \in \mathbf{Z}^{m \times n}$, and $b \in \mathbf{Z}^m$. Prove that, if $\bar{x}$ is not integral, then there exists $c \in \mathbf{Z}^n$ such that $\bar{x}$ is an optimal solution to $\max(c^T x : x \in P)$ and $c^T \bar{x}$ is not an integer.

(b) Do you need A and b to be integral?

Problem 6: [Bonus Problem] Prove Bland's Theorem that the Simplex Method always terminates if, among all choices of entering and leaving variable, we always choose the one with smallest subscript. (You may look in the literature, but write the proof in your own words.)