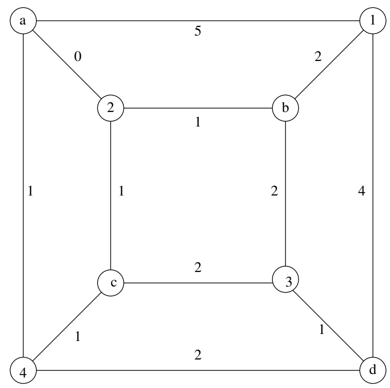
CO255 | **Assignment 5** | **Due: November 16**

Submit your assignment at the start of class. If a solution is not essentially correct you will get no credit. You may discuss assignment solutions with another student as long as neither of you has yet written a solution; taking written notes during the discussion is considered cheating.

Problem 1: Find a minimum cost perfect matching M in the graph below, and give a feasible dual solution $\bar{y}$ such that M is contained in $G^{=}(\bar{y})$. (You need not present all the details of the algorithm; it suffices to give M, $\bar{y}$, $\bar{c}$, and $G^{=}(\bar{y})$.)



Problem 2: Let $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ where $A \in \{0, \pm 1\}^{m \times n}$ be a totally unimodular matrix and $b \in \mathbf{Z}^m$.

(a) Let Z be the set of all integer-valued vectors in P. Prove that $\mathrm{conv}(Z) = P$. Hint: Its true for polytopes.

(b) Prove that, if b is $(0, \pm 1)$-valued, then the extreme points of A are $(0, \pm 1)$-valued.

Problem 3: Prove that, for any matrix $A \in \mathbf{R}^{X \times Y}$ there is a matrix $B \in \mathbf{Z}^{X \times Y}$ such that

- $|a_{xy} - b_{xy}| < 1$ for each $x \in X$ and $y \in Y$,
- $|(\sum_{x \in X} a_{xy}) - (\sum_{x \in X} b_{xy})| < 1$ for each $y \in Y$,
- $|(\sum_{y \in Y} a_{xy}) - (\sum_{y \in Y} b_{xy})| < 1$ for each $x \in X$, and
- $|(\sum_{x \in X} \sum_{y \in Y} a_{xy}) - (\sum_{x \in X} \sum_{y \in Y} b_{xy})| < 1$.

Hint: Introducing variables for the entries and for the various sums, construct an integral polytope P such that the integer vectors in P correspond to the desired matrices. Note that, for $a \in \mathbf{R}$ and $b \in \mathbf{Z}$, we have $|a - b| < 1$ if and only if $\lfloor a \rfloor \leq b \leq \lceil a \rceil$.

Problem 4: For a graph G, let $P = \{x \in \mathbf{R}^E : Ax = b,\ 0 \le x \le 2\}$ where A be the incidence matrix of a graph $G = (V, E)$ and $b \in \mathbf{Z}^V$ with all even entries. Prove that, if P is non-empty, then it has a feasible solution that is integer valued. (Hint: Reduce this to a problem on a bipartite graph that is twice as big.)

Problem 5: [Bonus Problem] Let $G = (V, E)$ be a bipartite graph with bipartition (X, Y) and let $b \in \mathbf{Z}^V$ such that $0 \le b_v \le \deg_G(v)$ for each $v \in V$. A *b-factor* of G is a subgraph H of G such that $\deg_H(v) = b_v$ for each $v \in V$. Use linear programming duality to prove that G has a b-factor if and only if

- $b(X) = b(Y)$, and
- for each $W \subseteq E$ and $Z \subseteq X$, $b(Z) \le |W| + b(N_{G-W}(Z))$.