CO255 **Assignment 6** **Due: November 30**

Submit your assignment at the start of class. If a solution is not essentially correct you will get no credit. You may discuss assignment solutions with another student as long as neither of you has yet written a solution; taking written notes during the discussion is considered cheating.

Problem 1: Let $S_1 = \{x \in \mathbf{R}^2 : x_2 \geq 2^{x_1}\}$, $S_2 = \text{Ball}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, 5\right)$, $S_3 = \text{Ball}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 4\right)$, $c = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, and $\bar{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Use the Cost Splitting Theorem to prove that $\bar{x}$ minimizes $(c^t x : x \in S_1 \cap S_2 \cap S_3)$.

Problem 2:

(a) Let $S \subseteq \mathbf{R}^n$ be a nonempty closed convex set and let $z \in \mathbf{R}^n$. Prove that there is a unique nearest point in S to z. (You need not reprove the existence of a nearest point.)

(b) Let $z, a, b \in \mathbf{R}^n$ be distinct points, and let $L = \text{conv}(\{a, b\})$. Prove that a is the closest point in L to z if and only if $(z - a)^T (b - a) \leq 0$.

(c) Let $S_1, S_2 \subseteq \mathbf{R}^n$ be disjoint, closed, convex, sets. Prove that, if $\inf(||x - y|| : x \in S_1, y \in S_2)$ is attained, then there exists $a \in \mathbf{R}^n$ and $b \in \mathbf{R}$ such that $a^t x \leq b$ for each $x \in S_1$ and $a^t x > b$ for all $y \in S_2$.

Problem 3: Let $S \subseteq \mathbf{R}^n$ be a closed convex set and let $\bar{x} \in S$ be in the boundary. Prove that there exists a point $z \in \mathbf{R}^n - S$ for which $\bar{x}$ is the nearest point in S to z. (Hint: Among all points $z \in \text{Ball}(\bar{x}, 1)$ consider one that is as far from S as possible.)

Problem 4:

(a) Give an example of two closed convex cones $K_1, K_1 \subseteq \mathbf{R}^n$ such that $K_1 + K_2$ is not closed.

(b) Let $K_1, K_1 \subseteq \mathbf{R}^n$ be convex cones. Prove that $K_1 + K_2$ is the smallest convex cone containing K_1 and K_2.

(c) Prove that, if $S_1, S_2 \subseteq \mathbf{R}^n$ are compact, then $S_1 + S_2$ is compact.

Problem 5:

(a) Let $S \subseteq \mathbf{R}^n$ be a convex set, $y \in \text{int}(S)$, let x be in the boundary of closure(S), and let $L = \text{conv}(\{x, y\})$. Prove that x is in the boundary of closure$(S \cap L)$

(b) Let $S_1, S_2 \subseteq \mathbf{R}^n$ with $\text{int}(S_1 \cap S_2) \neq \emptyset$. Prove that closure$(S_1 \cap S_2) =$ closure$(S_1) \cap$ closure(S_2).

Problem 6: [Bonus Problem] Prove that the Integer Programming Feasibility Problem is in $\mathcal{NP}$.