CO 330 Fall 2011 Midterm Exam.

6:00 - 7:50, Monday, Nov. 7th 2011.

There are five questions. Each question is worth six points.

1(a) *Prove the following binomial identity for all integers $n \geq 1$.*

$$\binom{2n}{n} = 2\sum_{j=0}^{n-1} \binom{n-1+j}{j}.$$

The LHS is the number of lattice paths from $(0,0)$ to (n,n). Each lattice path in the set $\mathcal{L}(n,n)$ uses exactly one of the edges $(n-1,j) \to (n,j)$ or $(j,n-1) \to (j,n)$ for some $0 \leq j \leq n-1$. The lattice paths in $\mathcal{L}(n,n)$ using the edge $(n-1,j) \to (n,j)$ are in bijection with the set of lattice paths from $(0,0)$ to $(n-1,j)$, so there are $|\mathcal{L}(n-1,j)| = \binom{n-1+j}{j}$ of these. The lattice paths in $\mathcal{L}(n,n)$ using the edge $(j,n-1) \to (j,n)$ are in bijection with the set of lattice paths from $(0,0)$ to $(j,n-1)$, so there are $|\mathcal{L}(j,n-1)| = \binom{j+n-1}{n-1} = \binom{n-1+j}{j}$ of these. Together, these observations prove the formula.

(b) *Which of the following is a correct q-binomial generalization of the formula in part (a)? (No explanation is needed.)*

(i) $$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = 2\sum_{j=0}^{n-1} \begin{bmatrix} n-1+j \\ j \end{bmatrix}_q$$

(ii) $$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{j=0}^{n-1} (q^{n-j} + q^{nj}) \begin{bmatrix} n-1+j \\ j \end{bmatrix}_q$$

(iii) $$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{j=0}^{n-1} (q^{j} + q^{n(n-j)}) \begin{bmatrix} n-1+j \\ j \end{bmatrix}_q$$

(iv) $$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{j=0}^{n-1} (q^{j} + q^{j(n-j)}) \begin{bmatrix} n-1+j \\ j \end{bmatrix}_q$$

Equation (iii) is correct, as one sees by keeping track of the areas of the lattice paths in the proof of part (a).

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2. *Consider a convex polygon with one edge marked with an arrow*

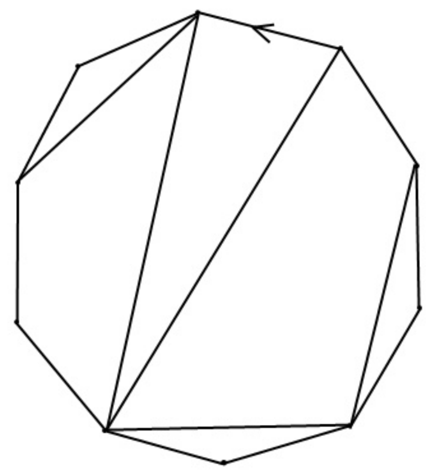

FIGURE 1. A dissection $\Delta \in \mathcal{K}$ with $p(\Delta) = 6$.

(to eliminate the dihedral symmetry). Let $\mathcal{K}$ be the set of all ways of dissecting such a polygon into triangles and quadrilaterals by drawing chords between some of the vertices. An example is shown in Figure 1. If Δ is such a dissection, then let $p(\Delta)$ be the number of triangles and quadrilaterals that it contains. So, for the example in Figure 1, $p(\Delta) = 6$.

(a) *Explain why the generating function $K(x) = \sum_{\Delta \in \mathcal{K}} x^{p(\Delta)}$ satisfies the functional equation*

$$K = x(1 + K)^2(2 + K).$$

Delete the edge of the polygon marked with the arrow (the "root edge"), and put arrows on the other edges of the triangle or quadrilateral that contains the root edge to form a directed path from the tail to the head of the root edge. The result is an ordered sequence of either two or three pieces – each piece is either a single edge marked with an arrow (for which $p = 0$) or a dissection in $\mathcal{K}$. Conversely, given a sequence of two or three such pieces, they can be stitched together to give a dissection in $\mathcal{K}$ with one more polygon (triangle or quadrilateral) than the total number of polygons in the pieces. That is, there is a bijection

$$\mathcal{K} \rightleftharpoons (\{\uparrow\} \cup \mathcal{K})^2 \cup (\{\uparrow\} \cup \mathcal{K})^3$$

in which $\uparrow$ denotes a single edge marked with an arrow. This yields the functional equation

$$K = x((1+K)^2 + (1+K)^3) = x(1+K)^2(2+K),$$

as claimed.

(b) *Use part (a) to show that for all $n \geq 1$, the number of $\Delta \in \mathcal{K}$ with $p(\Delta) = n$ is*

$$\frac{1}{n}\sum_{j=0}^{n-1}\binom{2n}{j}\binom{n}{j+1}2^{j+1}.$$

By LIFT, the number of $\Delta \in \mathcal{K}$ with $p(\Delta) = n$ is

$$\begin{aligned}
[x^n]K(x) &= \frac{1}{n}[u^{n-1}](1+u)^{2n}(2+u)^n \\
&= \frac{1}{n}[u^{n-1}]\sum_{j=0}^{2n}\binom{2n}{j}u^j\sum_{i=0}^{n}\binom{n}{i}2^{n-i}u^i \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\binom{2n}{j}\binom{n}{n-1-j}2^{n-(n-1-j)} \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\binom{2n}{j}\binom{n}{j+1}2^{j+1},
\end{aligned}$$

as claimed.

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3. *In a plane planted tree, a* middle child *is a node that is neither the leftmost child nor the rightmost child of its parent. Let $\mu(T)$ denote the number of middle children of the PPT T. Consider the two-variable generating function*

$$U(x,y) := \sum_{T\in\mathcal{U}} x^{n(T)}y^{\mu(T)},$$

in which the sum is over the set $\mathcal{U}$ of all PPTs.

(a) *Explain why $U(x,y)$ satisfies the functional equation*

$$U = x\left(1 + U + \frac{U^2}{1-yU}\right).$$

By the recursive structure of the set of PPTs, we have

$$\begin{aligned}
\mathcal{U} &\rightleftharpoons \{\odot\} \times \bigcup_{d=0}^{\infty} \mathcal{U}^d \\
T &\leftrightarrow (\odot, S_1, S_2, ..., S_d) \\
n(T) &= 1 + n(S_1) + n(S_2) + \cdots n(S_d) \\
\mu(T) &= (d-2)\chi[d \geq 2] + \mu(S_1) + \cdots + \mu(S_d)
\end{aligned}$$

In the expression for $\mu(T)$, the term $(d-2)\chi[d \geq 2]$ is the number of middle children of the root node: all other middle children are counted in exactly one of the subtrees S_1 to S_d. Thus the generating function satisfies

$$\begin{aligned}
U(x,y) &= x\left(1 + U(x,y) + \sum_{d=2}^{\infty} y^{d-2} U(x,y)^d\right) \\
&= x\left(1 + U + \frac{U^2}{1-yU}\right),
\end{aligned}$$

as claimed.

(b) *Use part (a) to show that the average value of $\mu(T)$ among all $\frac{1}{n}\binom{2n-2}{n-1}$ PPTs with n nodes is*

$$\overline{\mu}(n) = \frac{(n-2)(n-3)}{4n-6}.$$

By LIFT, the sum of $\mu(T)$ over all PPTs with n nodes is

$$\begin{aligned}
& [x^n] \left.\frac{\partial}{\partial y} U(x,y)\right|_{y=1} \\
&= \frac{1}{n}[u^{n-1}] \left.\frac{\partial}{\partial y}\left(1 + u + \frac{u^2}{1-yu}\right)^n\right|_{y=1} \\
&= [u^{n-1}]\left(\frac{1}{1-u}\right)^{n-1} \frac{u^2(-1)(-u)}{(1-u)^2} \\
&= [u^{n-1}]\frac{u^3}{(1-u)^{n+1}} = [u^{n-4}]\sum_{k=0}^{\infty}\binom{n+k}{n}u^k = \binom{2n-4}{n}.
\end{aligned}$$

Therefore, the average in question is

$$\begin{aligned}\overline{\mu}(n) &= \frac{\binom{2n-4}{n}}{\frac{1}{n}\binom{2n-2}{n-1}} \\ &= \frac{n(2n-4)!(n-1)!(n-1)!}{n!(n-4)!(2n-2)!} \\ &= \frac{(n-1)(n-2)(n-3)}{(2n-2)(2n-3)} = \frac{(n-2)(n-3)}{4n-6},\end{aligned}$$

as claimed.

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4. **(a)** *Let $\mathcal{A}$ be the set of partitions in which odd parts occur at most once each. Obtain a formula for the generating function $A(x) = \sum_{\lambda \in \mathcal{A}} x^{n(\lambda)}$.*

This is a direct application of Theorem 9.8: $M_j = \mathbb{N}$ if $j = 2i$ is even, while $M_j = \{0, 1\}$ of $j = 2i - 1$ is odd. Thus parts of size $2i$ contribute the factor

$$1 + x^{2i} + x^{4i} + x^{6i} + \cdots = \frac{1}{1 - x^{2i}}$$

while parts of size $2i - 1$ contribute the factor $1 + x^{2i-1}$. Therefore

$$A(x) = \prod_{i=1}^{\infty} \frac{1 + x^{2i-1}}{1 - x^{2i}}.$$

(b) *Let $\mathcal{B}$ be the set of partitions in which every even part is a multiple of 4. Explain why*

$$B(x) = \sum_{\lambda \in \mathcal{B}} x^{n(\lambda)} = \prod_{i=1}^{\infty} \frac{1}{(1 - x^{4i})(1 - x^{2i-1})}.$$

Again, this is a direct application of Theorem 9.8: we have $M_j = \mathbb{N}$ for all $j \equiv 0, 1, 3 \pmod 4$, but $M_j = \{0\}$ for $j \equiv 2 \pmod 4$. Thus

$$\begin{aligned}B(x) &= \prod_{j \equiv 0,1,3 \pmod 4} \frac{1}{1 - x^j} \\ &= \prod_{i=1}^{\infty} \frac{1}{(1 - x^{4i})(1 - x^{4i-3})(1 - x^{4i-1})} \\ &= \prod_{i=1}^{\infty} \frac{1}{(1 - x^{4i})(1 - x^{2i-1})},\end{aligned}$$

as claimed. In the last equality, we used the fact that

$$\prod_{i=1}^{\infty}(1-x^{4i-3})(1-x^{4i-1}) = \prod_{i=1}^{\infty}(1-x^{2i-1}).$$

(c) *Show that $A(x) = B(x)$. (Thus, for all $n \geq 0$ the number of partitions of size n in $\mathcal{A}$ equals the number of partitions of size n in $\mathcal{B}$.)*

Here we go!

$$\begin{aligned}
A(x) &= \prod_{i=1}^{\infty} \frac{1+x^{2i-1}}{1-x^{2i}} \\
&= \prod_{i=1}^{\infty} \frac{1+x^{2i-1}}{(1-x^i)(1+x^i)} \\
&= \prod_{i=1}^{\infty} \frac{1}{(1-x^i)(1+x^{2i})} \\
&= \prod_{i=1}^{\infty} \frac{1-x^{2i}}{(1-x^i)(1+x^{2i})(1-x^{2i})} \\
&= \prod_{i=1}^{\infty} \frac{1-x^{2i}}{(1-x^i)(1-x^{4i})} \\
&= \prod_{i=1}^{\infty} \frac{1}{(1-x^{2i-1})(1-x^{4i})} = B(x),
\end{aligned}$$

as claimed. In the last equality we used the fact that

$$\prod_{i=1}^{\infty} \frac{1-x^{2i}}{1-x^i} = \prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}}.$$

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5. *Let $f_k(x)$ for $k \geq 1$ be a sequence of formal power series in $R[[x]]$, in which R is a commutative ring. Assume that for every $J \in \mathbb{N}$ there exists a $K(J) \geq 1$ such that if $k \geq K(J)$ then the index of $f_k(x)$ is at least J: $I(f_k) \geq J$. Show that the infinite product*

$$\prod_{k=1}^{\infty}(1+f_k(x))$$

converges.

We must show that the limit

$$\lim_{L\to\infty}\prod_{k=1}^{L}(1+f_k(x))$$

exists. To do this we must show that for each $n\in\mathbb{N}$, the sequence of coefficients $[x^n]\prod_{k=1}^{L}(1+f_k(x))$ is eventually constant as $L\to\infty$. Consider any $n\in\mathbb{N}$, and let $K(n+1)$ be such that the index of $f_k(x)$ is at least $n+1$ for all $k\geq K(n+1)$. (By the assumption in the question, such a $K(n+1)$ exists.) So $f_k(x)=x^{n+1}g_k(x)$ for some power series $g_k(x)\in R[[x]]$, for each $k\geq K(n+1)$. Now, for all $L\geq K(n+1)$,

$$[x^n]\prod_{k=1}^{L}(1+f_k(x))=[x^n]]\prod_{k=1}^{K(n+1)-1}(1+f_k(x)),$$

since for $k\geq K(n+1)$ the factors $1+f_k(x)=1+x^{n+1}g_k(x)$ produce powers of x larger than x^n except when choosing the term 1 from each of them. Since the RHS is independent of $L\geq K(n+1)$ the sequence $[x^n]\prod_{k=1}^{L}(1+f_k(x))$ is eventually constant as $L\to\infty$. Thus, the limit exisits, and so

$$\prod_{k=1}^{\infty}(1+f_k(x))$$

converges.