CO 330 Fall 2011 Solutions #2

due Friday, Oct. 7.

Exercises: 3.3, 3.6, 3.8, 5.1, 5.3, 5.4, 5.7.

3.3. To prove the polynomial identity

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k}$$

it suffices to prove the numerical identity

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k b^{n-k}$$

for all natural numbers $a, b \in \mathbb{N}$. Let A be a set of size $|A| = a$ and let B be a set of size $|B| = b$, and assume that $A \cap B = \varnothing$. Then $A \cup B$ has size $a+b$, and by Example 1.7, the number of functions $f : N_n \to A \cup B$ is $(a+b)^n$. Consider such a function $f : N_n \to A \cup B$, and let $S = f^{-1}(A)$. This is a k-element subset of N_n, for some $0 \leq k \leq n$. Consider the functions $g = f|_S$ (f restricted to S) and $h = f|_{N_n \smallsetminus S}$ (f restricted to $N_n \smallsetminus S$). From the way that $S = f^{-1}(A)$ is defined it follows that $g : S \to A$ and $h : (N_n \smallsetminus S) \to B$. This construction $f \mapsto (S, g, h)$ determines a function from $\mathcal{F}(N_n, A \cup B)$ to

$$\bigcup_{S \in \mathcal{P}(N_n)} \{S\} \times \mathcal{F}(S, A) \times \mathcal{F}(N_n \smallsetminus S, B).$$

This is in fact a bijection: the inverse construction starts with (S, g, h) in which $S \subseteq N_n$, $g : S \to A$ and $h : (N_n \smallsetminus S) \to B$, and produces the function $\phi : N_n \to A \cup B$ defined by

$$\phi(i) = \begin{cases} g(i) & \text{if } i \in S, \\ h(i) & \text{if } i \notin S, \end{cases}$$

for all $i \in \mathcal{N}_n$. Thus we have a bijection

$$\mathcal{F}(N_n, A \cup B) \rightleftharpoons \bigcup_{S \in \mathcal{P}(N_n)} \{S\} \times \mathcal{F}(S, A) \times \mathcal{F}(N_n \smallsetminus S, B).$$

If $|S| = k$ then $|N_n \smallsetminus S| = n - k$, so that $|\mathcal{F}(S, A)| = a^k$ and $|\mathcal{F}(N_n \smallsetminus S, B) = b^{n-k}$. Since there are $\binom{n}{k}$ subsets of N_n of size k (for each

$0 \le k \le n$), by taking the cardinalities of the sets on both sides of the bijection, we obtain

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

as required. That completes the proof.

3.6. To prove the polynomial identity

$$\binom{x+1+n}{n} = \sum_{j=0}^{n} \binom{a+j}{j} \binom{x-a+n-j}{n-j}$$

it suffices to prove the numerical identity

$$\binom{b+1+n}{n} = \sum_{j=0}^{n} \binom{a+j}{j} \binom{b-a+n-j}{n-j}$$

for all integers b that are larger than or equal to a. So, fix a natural number $b \ge a$. The LHS is the number of lattice paths from $(0,0)$ to $(b+1,n)$; that is, $\#\mathcal{L}(b+1,n) = \binom{b+1+n}{n}$. Similarly, on the RHS $\binom{a+j}{j}$ is the number of lattice paths from $(0,0)$ to (a,j) for each $0 \le j \le n$. This is a starting point to figure out what the formula is saying. Eventually, we realize that every lattice path from $(0,0)$ to $(b+1,n)$ uses exactly one edge of the form $(a,j) \to (a+1,j)$ for some $0 \le j \le n$. Those paths P that use such an edge with a given value of j have the form $P = Q\mathsf{E}Q'$ in which $Q \in \mathcal{L}(a,j)$ and $Q' \in \mathcal{L}(b-a,n-j)$. Conversely, any such pair of lattice paths Q, Q' produces a lattice path $Q\mathsf{E}Q'$ from $(0,0)$ to $(b+1,n)$. In summary, we have a bijection

$$\mathcal{L}(b+1,n) \rightleftharpoons \bigcup_{j=0}^{n} \mathcal{L}(a,j) \times \{\mathsf{E}\} \times \mathcal{L}(b-a,n-j).$$

Taking the cardinality of each side yields the desired binomial identity.

3.8. To prove the polynomial identity

$$x^n = \sum_{k=0}^{n} k! S(n,k) \binom{x}{k},$$

it suffices to prove the numerical identity

$$a^n = \sum_{k=0}^{n} k! S(n,k) \binom{a}{k}$$

for all $a \in \mathbb{N}$. Let N be an n-element set, and let A be an a-element set. The LHS is the number of functions from N to A: $|\mathcal{F}(N, A)| = a^n$. Consider such a function $f : N \to A$. Define a set partition π_f of N by saying that $i, j \in N$ are in the same block of π_f if and only if $f(i) = f(j)$. There are $S(n, k)$ such set partitions with k blocks, by definition of the Stirling numbers (of the second kind). Given $\pi_f = \{B_1, ..., B_k\}$, we can define a function $g : \pi_f \to A$ by putting $g(B_i) = f(j)$ for any $j \in B_i$: from the way π_f is defined, this function $g : \pi_f \to A$ does not depend on the choices of $j \in B_i$ made in its definition. Also from the way that π_f is defined, the function $g : \pi_f \to A$ is an injection. The injection $g : \pi_f \to A$ is determined by its range $R \in \mathcal{B}(A, k)$ and a bijection $\sigma : \pi_f \to R$ from the domain to the range. Thus we have a construction that starts with $f : N \to A$ and produces a triple (π_f, R, σ).

Conversely, from a triple (π, S, τ) with π a set partition of N with k blocks (for some $0 \leq k \leq n$), $S \subseteq A$ a subset of A of size k, and $\tau : \pi \to S$ a bijection, we can construct a function $\varphi : N \to A$ by putting $\varphi(i) = \tau(B)$ where B is the unique block of π that contains $i \in N$. This defines a bijection between the set of all functions $f : N \to A$ and the set of all triples (π, S, τ) described above. (The set-theoretic notation for the set of all such triples is a bit cumbersome.)

The number k of blocks of π is in the range $0 \leq k \leq n$. If π has k blocks then there are $S(n, k)$ choices for π, $\binom{a}{k}$ choices for S, and $k!$ choices for τ. Thus,

$$a^n = \sum_{k=0}^{\infty} S(n, k) \binom{a}{k} k!,$$

as claimed.

5.1. This is the "q–analogue" of Example 3.3. The bijection at the heart of the matter is

$$\begin{aligned} \mathcal{L}(a+1, b) &\rightleftharpoons \bigcup_{j=0}^{b} \left(\mathcal{L}(a, j) \times \{\mathsf{EN}^{b-j}\} \right) \\ P &\leftrightarrow (Q, \mathsf{EN}^{b-j}) \\ \text{area}(P) &= \text{area}(Q) + (a+1)(b-j). \end{aligned}$$

(The last equation is easily seen by examining Figure 3.2 in the Course Notes.) It follows that

$$\begin{aligned}\left[\begin{array}{c} a+1+b \\ b \end{array}\right]_q &= \sum_{P\in\mathcal{L}(a+1,b)} q^{\mathrm{area}(P)} \\ &= \sum_{j=0}^{b} \sum_{Q\in\mathcal{L}(a,j)} q^{\mathrm{area}(Q)+(a+1)(b-j)} \\ &= \sum_{j=0}^{b} q^{(a+1)(b-j)} \left[\begin{array}{c} a+j \\ j \end{array}\right]_q .\end{aligned}$$

That does it!

5.3. This is the "q–analogue" of Exercise 3.5. Recall that the solution to Exercise 3.5 on HW#1 used subsets instead of lattice paths. We'll do the same for this question, using the fact that

$$q^{k(k+1)/2} \left[\begin{array}{c} n \\ k \end{array}\right]_q = \sum_{A\in\mathcal{B}(n,k)} q^{\mathrm{sum}(A)}.$$

(Here $\mathcal{B}(n,k)$ is the set of all k–element subsets of N_n.) The bijection at the heart of the matter is

$$\begin{aligned}\mathcal{B}(m+n,k) &\rightleftharpoons \bigcup_{j=0}^{k} (\mathcal{B}(m,j)\times\mathcal{B}(n,k-j)) \\ A &\leftrightarrow (S,T)\end{aligned}$$

To go from the LHS to the RHS we start with $A\in\mathcal{B}(m+n,k)$ and construct $S := A\cap N_m$ and

$$T := \{v\in N_n :\ v+m\in A\}.$$

Conversely, from (S,T) on the RHS we construct $A := S\cup\{v+m : v\in T\}$. These functions are mutually inverse bijections, as one can check. Keeping track of the sums of the sets in the bijections, we see that

$$\mathrm{sum}(A) = \mathrm{sum}(S) + \mathrm{sum}(T) + m\cdot(\#T).$$

Now we follow this information through the bijection...

$$\begin{aligned}
q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \sum_{A \in \mathcal{B}(n,k)} q^{\mathrm{sum}(A)} \\
&= \sum_{j=0}^{k} \sum_{S \in \mathcal{B}(m,j)} \sum_{T \in \mathcal{B}(n,k-j)} q^{\mathrm{sum}(S)+\mathrm{sum}(T)+m(k-j)} \\
&= \sum_{j=0}^{k} q^{m(k-j)} q^{j(j+1)/2} \begin{bmatrix} m \\ j \end{bmatrix}_q q^{(k-j)(k-j+1)/2} \begin{bmatrix} n \\ k-j \end{bmatrix}_q .
\end{aligned}$$

It remains to clean up the exponents of those extra factors of q. Multiply both sides by $q^{-k(k+1)/2}$. The exponent of q in the j–th term on the RHS is then

$$\begin{aligned}
& m(k-j) + j(j+1)/2 + (k-j)(k-j+1)/2 - k(k+1)/2 \\
= \; & m(k-j) + \frac{1}{2}\left[j^2 + j + k^2 - 2kj + j^2 + k - j - k^2 - k\right] \\
= \; & m(k-j) + j^2 - kj = (m-j)(k-j).
\end{aligned}$$

Therefore,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=0}^{k} q^{(m-j)(k-j)} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ k-j \end{bmatrix}_q ,$$

as was to be shown.

5.4. Let's continue from the solution to Exercise 3.6 given above. There is a bijection

$$\begin{aligned}
\mathcal{L}(b+1, n) &\rightleftharpoons \bigcup_{j=0}^{n} \mathcal{L}(a,j) \times \{\mathsf{E}\} \times \mathcal{L}(b-a, n-j) \\
P &\leftrightarrow (Q, \mathsf{E}, Q')
\end{aligned}$$

In this bijection, if $P \leftrightarrow (Q, \mathsf{E}, Q')$ and $Q \in \mathcal{L}(a,j)$, then

$$\mathrm{area}(P) = \mathrm{area}(Q) + (a+1)(n-j) + \mathrm{area}(Q').$$

(This is easy to see if you draw a picture.) It follows that

$$\begin{aligned}
\begin{bmatrix} b+1 \\ n \end{bmatrix}_q &= \sum_{P \in \mathcal{L}(b+1,n)} q^{\mathrm{area}(P)} \\
&= \sum_{j=0}^{n} \sum_{(Q,Q') \in \mathcal{L}(a,j) \times \mathcal{L}(b-a,n-j)} q^{\mathrm{area}(Q)+(a+1)(n-j)+\mathrm{area}(R)} \\
&= \sum_{j=0}^{n} q^{(a+1)(n-j)} \left(\sum_{Q} \in \mathcal{L}(a,j) q^{\mathrm{area}(Q)} \right) \left(\sum_{Q' \in \mathcal{L}(b-a,n-j)} q^{\mathrm{area}(Q')} \right) \\
&= \sum_{j=0}^{n} q^{(a+1)(n-j)} \begin{bmatrix} a+j \\ j \end{bmatrix}_q \begin{bmatrix} b-a+n-j \\ n-j \end{bmatrix}_q,
\end{aligned}$$

as was to be proved.

5.7. To prove that

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \sum_{P \in \mathcal{L}(a,b)} q^{\mathrm{area}(P)}$$

we use the bijection $\mathcal{L}(a,b) \rightleftharpoons \mathcal{B}(a+b,b)$ of Example 3.1 and the fact (Theorem 5.5), that

$$q^{b(b+1)/2} \begin{bmatrix} a+b \\ b \end{bmatrix}_q = \sum_{A \in \mathcal{B}(a+b,b)} q^{\mathrm{sum}(A)}.$$

If the lattice path P corresponds to the subset A in this bijection, then

$$\mathrm{area}(P) = \#E_A = \mathrm{sum}(A) - b(b+1)/2,$$

in which E_A is the set defined in Lemma 5.6. To see this, let $P = s_1 s_2 \ldots s_{a+b}$ in which each step s_i is either east E or north N. Then $A = \{i \in N_{a+b} : \; s_i = \mathsf{N}\}$. The area of P is the number of unit squares which have corners with integer coordinates and lie in the compact region enclosed by P and the line segments $(0,0) \to (0,b)$ and $(0,b) \to (a,b)$. Each such square $\square$ is directly above a unique E step of P, say $s_z = \mathsf{E}$, and directly to the left of a unique N step of P, say $s_a = \mathsf{N}$. Now this pair (a,z) has $a \in A$ and $z \notin A$ and $a > z$, so that $(a,z) \in E_A$. Conversely, from any pair $(a,z) \in E_A$ we can find the unit square $\square$ in $\mathbb{R}^2$ that is directly above the step $s_z = \mathsf{E}$ and directly left of the step $s_a = \mathsf{N}$. Thus, there is a bijection between the pairs in E_A and the unit squares comprising the area of P. This shows that area(P) = $\#E_A$, and suffices to prove the claim.