CO 330 Fall 2011 Solutions #3

due Friday, Oct. 21.

Exercises: 6.2, 6.3, 6.6, 6.8, 6.9, 7.2, 7.4, 7.6.

6.2. I will answer questions 6.1 and 6.2 together, as follows. For every $r \in \mathbb{N}$, and PPT $(T, \odot)$, let $c_r(T)$ denote the number of nodes of T that have exactly r children. Exercise 6.1 is the case $r = 0$, and Exercise 6.2 is the case $r = 1$. Consider the recursive structure of the set $\mathcal{U}$ of all PPTs:

$$\begin{aligned} \mathcal{U} &\rightleftharpoons \bigcup_{d=0}^{\infty} (\mathcal{U} \times \mathcal{U} \times \cdots \times \mathcal{U}) \quad [d \text{ factors}] \\ (T, \odot) &\leftrightarrow ((S_1, v_1), ..., (S_d, v_d)) \\ n(T) &= 1 + n(S_1) + n(S_2) + \cdots + n(S_d) \\ c_r(T) &= \chi[d = r] + c_r(S_1) + c_r(S_2) + \cdots + c_r(S_d) \end{aligned}$$

The equation for $n(T)$ was discussed in class. The equation for $c_r(T)$ uses the notation, for a proposition `P`,

$$\chi[\mathtt{P}] := \begin{cases} 1 & \text{if } \mathtt{P} \text{ is } \mathtt{TRUE}, \\ 0 & \text{if } \mathtt{P} \text{ is } \mathtt{FALSE}. \end{cases}$$

As before, on the RHS the first term is the contribution of the root node $\odot$, and the remaining terms are the contributions of the subtrees S_1 to S_d. Considering the two-variable generating function

$$U_r(x, y) := \sum_{(T, \odot) \in \mathcal{U}} x^{n(T)} y^{c_r(T)}$$

this leads to the functional equation

$$\begin{aligned} U_r(x, y) &= \sum_{d=0}^{\infty} x y^{\chi[d=r]} U_r(x, y)^d \\ &= \frac{x}{1 - U_r(x, y)} + (xy - x) U_r(x, y)^r. \end{aligned}$$

That is,

$$U(1 - U) = x[1 + (y - 1)(U^r - U^{r+1})].$$

When $r = 0$ or $r = 1$ this is a quadratic equation for $U = U_r(x, y)$, and we know how to solve it. (For $r \geq 2$ we will be able to use this equation once we establish the Lagrange Implicit Function Theorem.) For the moment I will deal with the cases $r = 0$ and $r = 1$ separately.

6.1 (The case $r = 0$.) We apply the Quadratic Formula:

$$\begin{aligned} U - U^2 &= x + x(y-1) - x(y-1)U \\ 0 &= U^2 - (xy - x + 1)U + xy \\ U &= \frac{(xy - x + 1) \pm \sqrt{(xy - x + 1)^2 - 4xy}}{2}. \end{aligned}$$

When $y = 1$ this must reduce to the one-variable generating function $U(x)$ for PPTs derived in class. Therefore

$$U_0(x, y) = \frac{xy - x + 1}{2} - \frac{1}{2}\sqrt{(xy - x + 1)^2 - 4xy}.$$

The sum of $c_0(T)$ over all $\frac{1}{n}\binom{2n-2}{n-1}$ PPTs with n nodes is

$$[x^n] \left. \frac{\partial}{\partial y} U_0(x, y) \right|_{y=0}.$$

Thus, we calculate that

$$\begin{aligned} \left. \frac{\partial U_0}{\partial y} \right|_{y=0} &= \frac{x}{2} - \frac{1}{2} \cdot \frac{1}{2}(1 - 4x)^{-1/2}[2(1)x - 4x] \\ &= \frac{x}{2}\left(1 + \frac{1}{\sqrt{1 - 4x}}\right) \\ &= x + \frac{1}{2}\sum_{n=2}^{\infty} \binom{2n-2}{n-1} x^n. \end{aligned}$$

When $n = 1$ the average value of $c_0(T)$ over all PPTs with one node is $1/1 = 1$. When $n \geq 2$ the average value of $c_0(T)$ over all PPTs with n nodes is $\frac{1}{2}\binom{2n-2}{n-1}/\frac{1}{n}\binom{2n-2}{n-1} = n/2$. (Check this in the case $n = 5$, illustrated in Figure 6.3.)

6.2. (The case $r = 1$.) We apply the Quadratic Formula:

$$\begin{aligned} U - U^2 &= x + x(y-1)U - x(y-1)U^2 \\ 0 &= (-xy + x + 1)U^2 + (xy - x - 1)U + x \\ U &= \frac{(-xy + x + 1) \pm \sqrt{(xy - x - 1)^2 - 4(-xy + x + 1)x}}{2(-xy + x + 1)} \\ U &= \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4x}{xy - x - 1}}. \end{aligned}$$

When $y = 1$ this must reduce to the one-variable generating function $U(x)$ for PPTs derived in class. Therefore

$$U_1(x,y) = \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4x}{xy - x - 1}}.$$

The sum of $c_1(T)$ over all $\frac{1}{n}\binom{2n-2}{n-1}$ PPTs with n nodes is

$$[x^n]\ \frac{\partial}{\partial y} U_1(x,y)\bigg|_{y=0}.$$

Thus, we calculate that

$$\begin{aligned} \frac{\partial U_1}{\partial y}\bigg|_{y=0} &= -\frac{1}{2}\cdot\frac{1}{2}(1-4x)^{-1/2}(4x(-1)(-1)^{-2}x) \\ &= \frac{x^2}{\sqrt{1-4x}} = \sum_{n=2}^{\infty}\binom{2n-4}{n-2}x^n. \end{aligned}$$

When $n = 1$ the average value of $c_1(T)$ over all PPTs with one node is $0/1 = 0$. When $n \geq 2$ the average value of $c_1(T)$ over all PPTs with n nodes is

$$\frac{\binom{2n-4}{n-2}}{\frac{1}{n}\binom{2n-2}{n-1}} = \frac{n(n-1)}{2(2n-3)}.$$

(Check this in the case $n = 5$, illustrated in Figure 6.3.)

6.3. For a SDLP P, let $p(P)$ denote the number of peaks of P. Consider the recursive structure of the set $\mathcal{V}$ of all SDLPs:

$$\begin{aligned} \mathcal{V} &\rightleftharpoons \bigcup_{r=0}^{\infty}(\mathsf{N}\mathcal{V}\mathsf{E})^r \\ P &\leftrightarrow (\mathsf{N}Q_1\mathsf{E}, \mathsf{N}Q_2\mathsf{E}, \ldots, \mathsf{N}Q_r\mathsf{E}) \\ n(P) &= (n(Q_1)+1) + \cdots + (n(Q_r)+1) \\ p(P) &= (p(Q_1) + \chi[Q_1 = \varepsilon]) + \cdots (p(Q_r) + \chi[Q_r = \varepsilon]) \end{aligned}$$

In the last line, ε is the empty string (of length 0) and for a proposition $\mathtt{P}$, $\chi[\mathtt{P}]$ is 1 if $\mathtt{P}$ is `TRUE` and 0 if $\mathtt{P}$ is `FALSE`. For the bivariate generating function

$$V(x,y) := \sum_{P\in\mathcal{V}} x^{n(P)}y^{p(P)}$$

this leads to the functional equation

$$V(x,y) = \sum_{r=0}^{\infty}(xV(x,y) - x + xy)^r = \frac{1}{1 + x - xy - xV(x,y)}.$$

This gives the quadratic equation

$$xV^2 + (xy - x - 1)V + 1 = 0$$

which has as solutions

$$V = \frac{1 + x - xy \pm \sqrt{(1 + x - xy)^2 - 4x}}{2x}.$$

Evaluated at $y = 1$, we must have $V(x, 1)$ being the one-variable generating function $V(x)$ derived in Chapter 6. Thus,

$$V(x, y) = \frac{1 + x - xy - \sqrt{(1 + x - xy)^2 - 4x}}{2x}.$$

Now

$$\begin{aligned}\left.\frac{\partial V(x,y)}{\partial y}\right|_{y=1} &= \frac{-x}{2x} - \frac{1}{2x} \cdot \frac{1}{2}(1-4x)^{-1/2}(2(1+x-x)^1(-x) \\ &= \frac{-1}{2} + \frac{1}{2}\frac{1}{\sqrt{1-4x}} \\ &= \sum_{n=1}^{\infty} \frac{1}{2}\binom{2n}{n} x^n.\end{aligned}$$

Since the number of SDLPs from $(0, 0)$ to (n, n) is $\frac{1}{n+1}\binom{2n}{n}$ for all $n \in \mathbb{N}$, it follows that the average number of peaks in a SDLP from $(0, 0)$ to (n, n) is 0 for $n = 0$, and is

$$\frac{\frac{1}{2}\binom{2n}{n}}{\frac{1}{n+1}\binom{2n}{n}} = \frac{n+1}{2}$$

for all $n \geq 1$.

6.6. The recursive structure for this set $\mathcal{A}$ of PPTs is as follows:

$$\begin{aligned}\mathcal{A} &\rightleftharpoons \{\odot\} \times (\mathcal{A}^0 \cup \mathcal{A}^1 \cup \mathcal{A}^2) \\ T &\leftrightarrow (\odot, S_1, .., S_d) \ \text{ with } \ d \in \{0, 1, 2\} \\ n(T) &= 1 + \sum_{i=1}^{d} n(S_i)\end{aligned}$$

This leads to the equation $A = x(1 + A + A^2)$ for $A(x) = \sum_{T\in\mathcal{A}} x^{n(T)}$. Applying the Quadratic Formula to $xA^2 + (x-1)A + x = 0$ yields

$$\begin{aligned} A(x) &= \frac{1 - x \pm \sqrt{(1-x)^2 - 4x^2}}{2x} \\ &= \frac{1-x}{2x}\left(1 \pm \sqrt{1 - 4\frac{x^2}{(1-x)^2}}\right) \end{aligned}$$

Applying the power series expansion of $\sqrt{1-4t}$ with $t = x^2/(1-x)^2$ yields

$$\begin{aligned} A(x) &= \frac{1-x}{2x}\left(1 \pm \left[1 - 2\sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k}\frac{x^{2k+2}}{(1-x)^{2k+2}}\right]\right) \\ &= \sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k}\frac{x^{2k+1}}{(1-x)^{2k+1}}, \end{aligned}$$

where we have had to take the minus sign in the $\pm$ to get a power series with nonnegative integer coefficients for $A(x)$. Continuing,

$$\begin{aligned} A(x) &= \sum_{k=0}^{\infty}\frac{1}{k+1}\binom{2k}{k}\sum_{j=0}^{\infty}\binom{2k+j}{2k}x^{2k+1+j} \\ &= \sum_{n=1}^{\infty}x^n\left[\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{2k+(n-1-2k)}{2k}.\right] \end{aligned}$$

Thus, for each $n \geq 1$, the number of PPTs with n nodes in $\mathcal{A}$ is

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{1}{k+1}\binom{2k}{k}\binom{n-1}{2k} = \sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{(n-1)!}{(k+1)!k!(n-1-2k)!},$$

as claimed.

6.8. By hand, one can check that $\tau_3 = 1$, $\tau_4 = 2$, $\tau_5 = 5$, and $\tau_6 = 14$. One then conjectures that the answer is the Catalan number $\tau_n = \frac{1}{n-1}\binom{2n-4}{n-2}$ for all $n \geq 3$. There are then two reasonable strategies for proving this formula – by generating functions or by finding a bijection. I'll do both.

I. Generating functions

To describe the recursive structure of the set of all triangulations, consider a triangulation Δ of an n-gon, with $n \geq 3$. The edge marked with an arrow ("root edge") is on exactly one triangle of Δ. Delete the root edge, and mark the other two edges of its triangle to give a directed path of length two from the tail of the root edge to the head of the root edge. The result is an ordered pair $(\Lambda, \mathcal{P})$ of triangulations: Λ is a triangulation of the j-gon containing the tail of the root edge of Δ, and $\mathcal{P}$ is a triangulation of the k-gon containing the head of the root edge of Δ. We have to admit the degenerate case that $j = 2$ or $k = 2$, so let us put $\tau_2 = 1$. (Luckily, this fits our conjecture as well!) Note that in this decomposition the sizes of the polygons are related by $j + k = n + 1$. The converse construction is apparent: for any ordered pair $(\Lambda, \mathcal{P})$ of triangulations with Λ triangulating a j-gon ($j \geq 2$) and $\mathcal{P}$ triangulating a k-gon ($k \geq 2$), we can stitch them together with a new root edge to obtain a triangulation of an n-gon, where $n = j + k - 1$. (Note that if n and j are given, then $k = n + 1 - j$.) Thus, we have established a bijection which shows that for all $n \geq 3$,

$$\tau_n = \sum_{j=2}^{n-1} \tau_j \tau_{n+1-j}.$$

Together with the base case $\tau_2 = 1$ this determines the sequence $(\tau_n : n \geq 2)$. You can check that it agrees with the data for n up to 6.

To solve this recurrence, consider the generating function

$$T(x) = \sum_{n=2}^{\infty} \tau_n x^n = \sum_{\Delta} x^{n(\Delta)}.$$

(In the second summation, Δ ranges over all triangulations of all polygons, and Δ triangulates an $n(\Delta)$-gon.) We calculate that

$$\begin{aligned} T(x) &= \sum_{n=2}^{\infty} \tau_n x^n \\ &= \tau_2 x^2 + \sum_{n=3}^{\infty} \left(\sum_{j=2}^{n-1} \tau_j \tau_{n+1-j} \right) x^n \\ &= x^2 + \sum_{j=2}^{\infty} \tau_j \sum_{k=2}^{\infty} \tau_k \, x^{j+k-1} \\ &= x^2 + T(x)^2/x. \end{aligned}$$

That is, the generating function $T(x)$ satisfies the functional equation $T^2/x - T + x^2 = 0$, which we solve by the Quadratic Formula and the

Binomial Series expansion, to yield

$$\begin{aligned} T(x) &= \frac{1 \pm \sqrt{1-4x}}{2/x} \\ &= \frac{x}{2} \pm \frac{x}{2}\left(1 - 2\sum_{h=1}^{\infty} \frac{1}{h}\binom{2h-2}{h-1}x^h\right) \\ &= \sum_{h=1}^{\infty} \frac{1}{h}\binom{2h-2}{h-1}x^{h+1} \\ &= \sum_{n=2}^{\infty} \frac{1}{n-1}\binom{2n-4}{n-2}x^n. \end{aligned}$$

(In the penultimate equality we use the fact that the coefficients are nonnegative to determine the choice of the $\pm$ sign.) That's it!

II. Bijection

Here is a bijection between the set of all triangulations and the set of all binary rooted trees. In this bijection, if a triangulation Δ corresponds to a BRT T, then $n(\Delta) = n(T) + 2$. Since we have enumerated BRTs, this shows that there are $\frac{1}{n-1}\binom{2n-4}{n-2}$ triangulations of an n-gon, for each $n \geq 3$.

To define the bijection, start with a triangulation Δ. Put a node in the middle of each triangle – the node in the triangle containing the root edge of Δ will be the root node of T. Draw edges between nodes of T across the edges of Δ. Each node has at most two children, since it sits in a triangle of Δ and (except for the root node) one of its neighbours is its parent. Each child w of each node v is labelled left or right according to which edge of the triangle containing v is crossed when going from v to w: one enters the triangle containing v coming from its parent (or the root edge, in case v is the root node) and then crosses either the left or the right edge of the triangle to get to w. Thus, T is a BRT. The number of nodes of T is the number of triangles of Δ, which is $n(\Delta) - 2$. (The converse construction of Δ given T is left to you to explain.)

6.9. For each $n \geq 1$, let $\mathcal{Y}_n$ be the set of 2-by-n Standard Young tableaux, and let $\mathcal{V}_n$ be the set of SDLPs from $(0,0)$ to (n,n). By Theorem 6.9, $\#\mathcal{V}_n = \frac{1}{n+1}\binom{2n}{n}$. To answer the question it suffices to find a bijection $\mathcal{Y}_n \rightleftharpoons \mathcal{V}_n$.

Let $A = (a_{ij})$ be a 2-by-n SYT. Define a sequence $P = s_1 s_2 ... s_{2n}$ as follows: each $s_i \in \{\mathsf{N}, \mathsf{E}\}$, and

$$s_i := \begin{cases} \mathsf{N} & \text{if } i \text{ is in row one of } A, \\ \mathsf{E} & \text{if } i \text{ is in row two of } A. \end{cases}$$

The left-to-right and top-to-bottom increasing condition in the definition of SYT shows that for each $1 \le k \le 2n$, there are at least as many Ns as there are Es in the subsequence $s_1 ... s_k$. By Lemma 6.11 and Example 6.12, it follows that P is a SDLP.

Conversely, given a SDLP $P = s_1 ... s_{2n}$ construct a 2-by-n SYT as follows. Start with an empty 2-by-n array A of cells. As k goes from 1 to $2n$ (increasing by 1 each step), put the number k in the leftmost empty cell in the first row of A if $s_i = \mathsf{N}$, or in the leftmost empty cell in the second row of A if $s_i = \mathsf{E}$. From Example 6.12 and Lemma 6.11 it is not too hard to see that the increasing conditions defining a SYT are satisfied by the result.

One sees after a while that composing these constructions in either order yields the identity functions $\mathcal{Y}_n \to \mathcal{Y}_n$ and $\mathcal{V}_n \to \mathcal{V}_n$. Therefore, these constructions are mutually inverse bijections, completing the proof.

7.2. **(a)** Assume that R is an integral domain. Suppose that $R[x]$ is not an integral domain, and let $p(x) = \sum_{i=0}^{n} a_i x^i$ and $q(x) = \sum_{j=0}^{m} b_j x^j$ be nonzero polynomials in $R[x]$ such that $p(x)q(x) = 0$. We may assume that p has degree n and q has degree m, so that $a_n \neq 0$ and $b_m \neq 0$ are nonzero elements of R. Taking the coefficient of x^{n+m} on both sides of the equation $p(x)q(x) = 0$, we see that $a_n b_m = 0$. This shows that R contains zero-divisors, contradicting the assumption that R is an integral domain.

(b) Note that $1/x$ is in $R(x)$ but not in $R[[x]]$, as is clearly seen. Also, $(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$ is in $R[[x]]$ but not in $R(x)$. To see this, if it were a rational function then $(1-4x)^{-1/2} = p(x)/q(x)$ for two polynomials, with $q(x) \neq 0$. But then $q(x)^2 = (1-4x)p(x)^2$. The LHS is a polynomial of even degree, while the RHS is a polynomial of odd degree. This contradiction shows that $(1-4x)^{-1/2}$ is not in $R(x)$.

(c) Assume that R is a field, and let $p(x)/q(x)$ be a quotient of polynomials, with $q(x) \neq 0$. If k is the smallest power of x that occurs with nonzero coefficient in $q(x)$ then we can write $q(x) = c_k x^k f(x)$ for some polynomial $f(x)$ with constant term equal to one, and $c_k \neq 0$. By Proposition 7.5, $f(x)$ is invertible in $R[[x]]$, which is a subset of $R((x))$, and it follows that $q(x)^{-1} = c_k^{-1}x^{-k}f(x)^{-1}$ in $R((x))$. Thus,

$$\frac{p(x)}{q(x)} = p(x)q(x)^{-1} = p(x)c_k^{-1}x^{-k}f^{-1}(x)$$

is a formal Laurent series in $R((x))$. Thus, $R(x)$ is a subset of $R((x))$.
(d) Note that $1/2$ is in $\mathbb{Z}(x)$ but not in $\mathbb{Z}((x))$. (From part (b) it follows that $\mathbb{Z}(x)$ is contained in $\mathbb{Q}((x))$.)
(e) The ring $R[[x]][y]$ consists of polynomials in y whose coefficients are power series in $R[[x]]$: that is, something of the form $\sum_{i=0}^{n} a_i(x)y^i$. The ring $R[y][[x]]$ consists of power series in x whose coefficients are polynomials in $R[y]$: that is, something of the form $\sum_{j=0}^{\infty} b_j(y)x^j$. Writing an element of the form $\sum_{i=0}^{n} a_i(x)y^i$ in terms of powers of x, we see that it has the form $\sum_{j=0}^{\infty} b_j(y)x^j$ and is such that $\deg(b_j) \leq n$ for all $j \in \mathbb{N}$. Thus $R[[x]][y]$ is contained in $R[y][[x]]$. The element

$$\frac{1}{1-(y+1)x} = \sum_{j=0}^{\infty}(1+y)^j x^j$$

is an element of $R[y][[x]]$ that is not contained in $R[[x]][y]$.

7.4. To begin with, the product rule is straightforward for powers of x: for any integers $m, n \in \mathbb{Z}$:

$$\begin{aligned}\frac{d}{dx}(x^m \cdot x^n) &= (m+n)x^{m+n-1} \\ &= (mx^{m-1})x^n + x^m(nx^{n-1}) \\ &= \left(\frac{d}{dx}x^m\right)x^n + x^m\left(\frac{d}{dx}x^n\right).\end{aligned}$$

Now, since d/dx is a linear operator, it follows that for any two formal Laurent series $f(x) = \sum_{m=I(f)}^{\infty} a_m x^m$ and $g(x) = \sum_{n=I(g)}^{\infty} b_n x^n$ we have

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \frac{d}{dx} \sum_{m=I(f)}^{\infty} \sum_{n=I(g)}^{\infty} a_m b_n x^{m+n} \\
&= \sum_{m=I(f)}^{\infty} \sum_{n=I(g)}^{\infty} a_m b_n \left[(mx^{m-1})x^n + x^m(nx^{n-1})\right] \\
&= \left[\sum_{m=I(f)}^{\infty} m a_m x^{m-1} \sum_{n=I(g)}^{\infty} b_n x^n\right] + \left[\sum_{m=I(f)}^{\infty} a_m x^m \sum_{n=I(g)}^{\infty} n b_n x^{n-1}\right] \\
&= \left(\frac{d}{dx} f(x)\right) g(x) + f(x) \left(\frac{d}{dx} g(x)\right).
\end{aligned}$$

That's it!

7.6. The sequence of formal power series is defined by $f_0(x) := 1$, $f_1(x) := 1$, and $f_{k+1}(x) := f_k(x) + x^k f_{k-1}(x)$ for all $k \geq 1$. This sequence of formal power series converges. (Writing out the first few, up to $f_7(x)$, say, is a good example.) To check the definition of convergence, consider the sequence of coefficients of x^n , for any $n \in \mathbb{N}$. We must show that this sequence of coefficients is eventually constant. That is, we must show that there is an index K_n and a value A_n such that for all $k \geq K_n$, $[x^n]f_k(x) = A_n$. Now, from the defining recurrence, note that if $k \geq n+1$ then

$$[x^n]f_{k+1}(x) = [x^n]f_k(x) + [x^n]x^k f_{k-1}(x) = [x^n]f_k(x).$$

Thus, for all $k \geq n+1$ it follows by induction on k that

$$[x^n]f_k(x) = [x^n]f_{n+1}(x).$$

This verifies the definition of convergence, but the identity of the limiting series $f(x) = \lim_{k\to\infty} f_k(x)$ remains mysterious.
