CO 330 Fall 2011 Solutions #4

due Friday, Nov.4.

Exercises: 7.8(a), 7.11, 7.12(a,b,c), 8.1, 8.4, 8.6, 9.1, 9.3.

7.8(a) By the Product Rule (Exercise 7.4), for any power series $f(x)$ and $g(x)$ in $R[[x]]$ we have

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

We begin by proving that $\frac{d}{dx}g(x)^n = ng(x)^{n-1}g'(x)$ for all $n \in \mathbb{N}$. When $n = 0$ we have $g(x)^0 = 1$ on the LHS, and (even if $g(x)$ is not invertible) the RHS is zero, so the formula holds. Also, when $n = 1$ the formula states that $\frac{d}{dx}g(x) = g'(x)$, which is true. This is the basis of induction. Proceeding to the induction step, consider $g(x)^{n+1} = g(x)^n \cdot g(x)$. Applying the Product Rule (Exercise 7.4) and the induction hypothesis, we calculate that

$$\begin{aligned}\frac{d}{dx}g(x)^{n+1} &= (ng(x)^{n-1}g'(x))g(x) + g(x)^n g'(x)\\ &= (n+1)g(x)^n g'(x),\end{aligned}$$

as required. This finishes the induction and establishes the claim.

Now, consider any power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ in $R[[x]]$. By linearity of the differentiation operator and the claim above, we calculate that

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \sum_{n=0}^{\infty} a_n \frac{d}{dx}g(x)^n\\ &= \sum_{n=0}^{\infty} a_n n g(x)^{n-1}g'(x) = f'(g(x))g'(x)\end{aligned}$$

since $f'(x) = \sum_{n=0}^{\infty} na_n x^{n-1}$. If either $f(x)$ is a polynomial or the constant term of $g(x)$ is zero then everything in sight is a well-defined formal power series. That's it!

7.11. First, note that

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \frac{[a+b]!_q}{[a]!_q[b]!_q} = \frac{[a+b]_q[a+b-1]_q\cdots[a+2]_q[a+1]_q}{[b]_q[b-1]_q\cdots[2]_q[1]_q}.$$

Now, for any $j \in \mathbb{N}$,

$$\begin{aligned} \frac{[a+j]_q}{[j]_q} &= \frac{1+q+q^2+\cdots+q^{a+j-1}}{1+q+q^2+\cdots q^{j-1}} \\ &= \frac{(1+q+q^2+\cdots+q^{a+j-1})(1-q)}{(1+q+q^2+\cdots q^{j-1})(1-q)} \\ &= \frac{1-q^{a+j}}{1-q^j}. \end{aligned}$$

Therefore,

$$\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \prod_{j=1}^{b}\left(\frac{1-q^{a+j}}{1-q^j}\right).$$

For any power of q, say q^n, once $a \geq n$ the sequence of coefficients of q^n in $\begin{bmatrix} a+b \\ b \end{bmatrix}_q$ becomes constant, agreeing with the coefficient of q^n in $\prod_{j=1}^{b}(1-q^j)^{-1}$. Therefore,

$$\lim_{a\to\infty}\begin{bmatrix} a+b \\ b \end{bmatrix}_q = \prod_{j=1}^{b}\frac{1}{1-q^j}.$$

7.12(a) From Example 7.9(a) we have

$$\log\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty}\frac{x^k}{k}.$$

Let's use this by writing

$$f(x) = \frac{1}{1-g(x)}$$

so that

$$\log(f(x)) = \log\left(\frac{1}{1-g(x)}\right) = \sum_{k=1}^{\infty}\frac{g(x)^k}{k}.$$

The issue is whether the limit

$$\lim_{K\to\infty}\sum_{k=1}^{K}\frac{g(x)^k}{k}$$

exists. But from $f(x) = (1-g(x))^{-1}$, we get $f(x)^{-1} = 1-g(x)$, so that $g(x) = 1 - f(x)^{-1}$. Since $[x^0]f(x) = 1$ it follows that $[x^0]f(x)^{-1} = 1$ as well, and hence that $[x^0]g(x) = 0$. This is enough to ensure that the above limit exists, by Proposition 7.15(iii).

7.12(b) One must be a little bit careful, because $\log(x)$ is not a power series in x. Instead, one must use $\log((1-x)^{-1})$. Note that

$$\frac{d}{dx}\log\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{k=1}^{\infty}\frac{x^k}{k} = \sum_{k=1}^{\infty}x^{k-1} = \frac{1}{1-x}.$$

Now, write $f(x) = (1-g(x))^{-1}$ as in part (a), and use the Chain Rule (Exercise 7.8(a)) to see that

$$f'(x) = (-1)(1-g(x))^{-2}(-g'(x)),$$

so that

$$g'(x) = (1-g(x))^2 f'(x).$$

Finally, we calculate using the Chain Rule again, that

$$\begin{aligned}\frac{d}{dx}\log(f(x)) &= \frac{d}{dx}\log\left(\frac{1}{1-g(x)}\right)\\ &= \frac{1}{1-g(x)}g'(x) = \frac{(1-g(x))^2}{1-g(x)}f'(x)\\ &= f(x)^{-1}f'(x),\end{aligned}$$

as required.

7.12(c) From part (a), $\log(\exp(x)) = \sum_{n=0}^{\infty} c_n x^n$ is a well-defined formal power series. We determine the coefficients c_n by applying part (b). Note that $\frac{d}{dx}\exp(x) = \exp(x)$, so that

$$\frac{d}{dx}\log(\exp(x)) = \exp(x)^{-1}\exp(x) = 1.$$

That is,

$$1 = \frac{d}{dx}\sum_{n=0}^{\infty}c_n x^n = \sum_{n=0}^{\infty}nc_n x^{n-1}.$$

From this we see that $c_1 = 1$ and $c_n = 0$ for all $n \geq 2$. This gives no information about c_0, however. But from the argument for part (a) we see that $[x^0]\log(f(x)) = 0$ for any formal power series $f(x)$ with $[x^0]f(x) = 1$. Therefore, $c_0 = 0$ as well, and so

$$\log(\exp(x)) = x,$$

as required.

8.1. Fix a positive integer c. Let $\mathcal{Q}$ be the set of plane planted trees (PPTs) in which the number of children of each node is a multiple of c. By deleting the root node of a tree in $\mathcal{Q}$ we obtain an ordered sequence of substrees in $\mathcal{Q}$. The number of subtrees must be a multiple of c (maybe zero). Thus, we have a bijection

$$\mathcal{Q} \rightleftharpoons \{\odot\} \times \bigcup_{m=0}^{\infty} \mathcal{Q}^{mc}.$$

The number of nodes in the original tree is one (for the root node) plus the total number of nodes in all the subtrees. This leads to the following functional equation for the generating function $Q(x) = \sum_{T\in\mathcal{Q}} x^{n(T)}$:

$$Q(x) = x \sum_{m=0}^{\infty} Q(x)^{mc} = \frac{x}{1 - Q(x)^c}.$$

The number of PPTs in $\mathcal{Q}$ with n nodes is $[x^n]Q(x)$. We can calculate this using LIFT with $G(u) = 1/(1-u^c)$, by the form of the functional equation for $Q(x)$. For $n = 0$, $[x^0]Q(x) = 0$, and for all $n \geq 1$,

$$\begin{aligned} [x^n]Q(x) &= \frac{1}{n}[u^{n-1}]\left(\frac{1}{1-u^c}\right)^n \\ &= \frac{1}{n}[u^{n-1}]\sum_{j=0}^{\infty}\binom{n-1+j}{n-1}u^{jc} \\ &= \begin{cases} \frac{1}{n}\binom{n-1+j}{n-1} & \text{if } n = jc+1 \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

That's it!

8.4. Let $\mathcal{U}$ be the set of plane planted trees (PPTs) and for $T \in \mathcal{U}$ let $f(T)$ be the number of nodes of T with at least three children. From the recursive structure of $\mathcal{U}$ we have

$$\begin{aligned} \mathcal{U} &\rightleftharpoons \{\odot\} \times \bigcup_{d=0}^{\infty} \mathcal{U}^d \\ T &\leftrightarrow (\odot, S_1, S_2, ..., S_d) \\ n(T) &= 1 + n(S_1) + n(S_2) + \cdots + n(S_d) \\ f(T) &= \chi[d \geq 3] + f(S_1) + f(S_2) + \cdots + f(S_d) \end{aligned}$$

where, as ususal, $\chi[\mathtt{P}] = 1$ if P is **true**, and $\chi[\mathtt{P}] = 0$ if P is **false**. Thus, the generating function

$$U(x, y) = \sum_{T \in \mathfrak{U}} x^{n(T)} y^{f(T)}$$

satisfies the functional equation

$$\begin{aligned} U &= x(1 + U + U^2 + y(U^3 + U^4 + \cdots)) \\ &= x\left(\frac{1}{1-U} + (y-1)\frac{U^3}{1-U}\right) \\ &= x\left(\frac{1 + (y-1)U^3}{1-U}\right). \end{aligned}$$

LIFT applies here with $F(u) = u$ and $G(u) = (1 + (y-1)u^3)/(1-u)$. The number of PPTs with $n \geq 1$ nodes is $\frac{1}{n}\binom{2n-2}{n-1}$, by Theorem 6.7. The sum of $f(T)$ over all PPTs with $n \geq 4$ nodes is

$$\begin{aligned} & [x^n] \left.\frac{\partial}{\partial y} U(x, y)\right|_{y=1} \\ = & \left.\frac{\partial}{\partial y}[x^n]U(x, y)\right|_{y=1} \\ = & \left.\frac{\partial}{\partial y}\frac{1}{n}[u^{n-1}]\left(\frac{1 + (y-1)u^3}{1-u}\right)^n\right|_{y=1} \\ = & [u^{n-1}]\frac{u^3}{(1-u)^n} \\ = & [u^{n-4}]\sum_{j=0}^{\infty}\binom{n-1+j}{n-1}u^j = \binom{2n-5}{n-1}. \end{aligned}$$

Therefore, the average value of $f(T)$ over all PPTs with $n \geq 4$ nodes is

$$\begin{aligned} \overline{f}(n) &= \frac{\binom{2n-5}{n-1}}{\frac{1}{n}\binom{2n-2}{n-1}} \\ &= \frac{n \cdot (2n-5)! \cdot (n-1)! \cdot (n-1)!}{(n-1)! \cdot (n-4)! \cdot (2n-2)!} \\ &= \frac{n(n-1)(n-2)(n-3)}{(2n-2)(2n-3)(2n-4)} = \frac{n(n-3)}{4(2n-3)} = \frac{n^2 - 3n}{8n - 12}, \end{aligned}$$

as claimed.

8.6. Let $\mathcal{U}$ be the set of plane planted trees (PPTs) and for $T \in \mathcal{U}$ let $d(T)$ be the degree of the root node of T. From the recursive structure of $\mathcal{U}$ we have

$$\begin{aligned}
\mathcal{U} &\rightleftharpoons \{\odot\} \times \bigcup_{d=0}^{\infty} \mathcal{U}^d \\
T &\leftrightarrow (\odot, S_1, S_2, ..., S_d) \\
n(T) &= 1 + n(S_1) + n(S_2) + \cdots + n(S_d) \\
d(T) &= d
\end{aligned}$$

Thus, the generating function

$$U = U(x, y) = \sum_{T \in \mathcal{U}} x^{n(T)} y^{d(T)}$$

satisfies the functional equation

$$U = x(1 + yR + y^2R^2 + \cdots) = x\frac{1}{1 - yR}$$

in which $R = R(x) = U(x, 1)$. Thus, R satisfies the functional equation

$$R = x\frac{1}{1 - R},$$

which is the familiar recursion for PPTs from Theorem 6.7. LIFT applies here with $F(u) = 1/(1 - yu)$ and $G(u) = 1/(1 - u)$. (One has to be careful about the "exponent shift" in the following calculation, since $U = xF(R)$.) Note that $F'(u) = y/(1 - yu)^2$. The sum of $d(T)$

over all PPTs with $n \geq 2$ nodes is

$$\begin{aligned}
& [x^n]\, \frac{\partial}{\partial y} U(x,y) \Big|_{y=1} \\
= \;& \frac{\partial}{\partial y} [x^n] x F(R(x)) \Big|_{y=1} \\
= \;& \frac{\partial}{\partial y} \frac{1}{n-1} [u^{n-2}] F'(u) G(u)^{n-1} \Big|_{y=1} \\
= \;& \frac{\partial}{\partial y} \frac{1}{n-1} [u^{n-2}] \frac{y}{(1-yu)^2 (1-u)^{n-1}} \Big|_{y=1} \\
= \;& \frac{1}{n-1} [u^{n-2}] \frac{1}{(1-u)^{n-1}} \left(\frac{1}{(1-u)^2} + \frac{2u}{(1-u)^3} \right) \\
= \;& \frac{1}{n-1} [u^{n-2}] \frac{1+u}{(1-u)^{n+2}} \\
= \;& \frac{1}{n-1} \left(\binom{2n-1}{n+1} + \binom{2n-2}{n+1} \right)
\end{aligned}$$

Thus, the average of $d(T)$ among all $\frac{1}{n}\binom{2n-2}{n-1}$ PPTs with $n \geq 2$ nodes is

$$\begin{aligned}
\overline{d}(n) &= \frac{\frac{1}{n-1}\left(\binom{2n-1}{n+1} + \binom{2n-2}{n+1}\right)}{\frac{1}{n}\binom{2n-2}{n-1}} \\
&= \frac{n}{n-1} \cdot \frac{(n-1)!(n-1)!}{(2n-2)!} \left(\frac{(2n-1)!}{(n+1)!(n-2)!} + \frac{(2n-2)!(n-2)}{(n+1)!(n-3)!(n-2)} \right) \\
&= \frac{n(n-1)!(n-1)!(2n-2)!}{(n-1)(2n-2)!(n+1)!(n-2)!} (2n-1+n-2) \\
&= \frac{3n-3}{n+1},
\end{aligned}$$

as claimed.

9.1. These are all direct applications of Theorem 9.8.

9.1(a)

$$\prod_{i=1}^{\infty} \left(\frac{1}{1 - x^{2i} y} \right)$$

9.1(b)

$$\prod_{i=1}^{\infty}\left(\frac{1+x^{2i-1}y}{1-x^{2i}y}\right)$$

9.1(c)

$$\prod_{i=1}^{\infty}\left(\frac{1}{(1-x^{4i}y)(1-x^{4i-2}y^2)}\right)$$

9.1(d)

$$\prod_{i=1}^{\infty}\left(\frac{1+x^{2i-1}y-x^{4i-2}y^2}{(1-x^{4i}y^2)(1-x^{4i-2}y^2)}\right)$$

9.3(a) Each part occurs at most three times, so $M_j = \{0,1,2,3\}$ for all $j \geq 1$, using the notation of Theorem 9.8. The generating function for this set $\mathcal{A}$ keeping track of $x^{n(\lambda)}$ is

$$\Phi_{\mathcal{A}}(x) = \sum_{\lambda\in\mathcal{A}} x^{n(\lambda)} = \prod_{j=1}^{\infty}\left(1+x^j+x^{2j}+x^{3j}\right) = \prod_{j=1}^{\infty}\left(\frac{1-x^{4j}}{1-x^j}\right).$$

9.3(b) Each even part occurs at most once, so $M_j = \{0,1\}$ if $j = 2i$ is even, and $M_j = \mathbb{N}$ if $j = 2i-1$ is odd, for all $i \geq 1$. The generating function for this set $\mathcal{B}$ keeping track of $x^{n(\lambda)}$ is

$$\Phi_{\mathcal{B}}(x) = \sum_{\lambda\in\mathcal{B}} x^{n(\lambda)} = \prod_{i=1}^{\infty}\frac{1+x^{2i}}{1-x^{2i-1}}.$$

9.3(c) To see that there are the same number of partitions of size n in $\mathcal{A}$ as in $\mathcal{B}$ (for all $n \in \mathbb{N}$) it suffices to prove that $\Phi_{\mathcal{A}}(x) = \Phi_{\mathcal{B}}(x)$. Here we go:

$$\begin{aligned}\Phi_{\mathcal{A}}(x) &= \prod_{j=1}^{\infty}\left(\frac{1-x^{4j}}{1-x^j}\right)\\ &= \prod_{j=1}^{\infty}\left(\frac{(1-x^{2j})(1+x^{2j})}{1-x^j}\right)\\ &= \prod_{j=1}^{\infty}\left(\frac{1-x^{2j}}{1-x^{2j-1}}\right)\\ &= \Phi_{\mathcal{B}}(x).\end{aligned}$$

Done!
