CO 330 Fall 2011 Solutions #5

due Friday, Nov.18.

Exercises: 9.4, 9.6, 9.8, 10.3, 10.4, 10.11.

9.4
(a)

$$A(x) = \prod_{j=1}^{\infty}(1 + x^j + x^{4j} + x^{5j}).$$

(b)

$$B(x) = \prod_{i=1}^{\infty}\frac{1 + x^{4i}}{1 - x^{2i-1}}.$$

(c)

$$\begin{aligned} A(x) &= \prod_{j=1}^{\infty}(1 + x^j)(1 + x^{4j}) \\ &= \prod_{j=1}^{\infty}(1 + x^j)(1 + x^{4j})\frac{1 - x^j}{1 - x^j} \\ &= \prod_{j=1}^{\infty}\frac{(1 - x^{2j})(1 + x^{4j}}{1 - x^j} \\ &= \prod_{j=1}^{\infty}\frac{1 + x^{4j}}{1 - x^{2j-1}} = B(x). \end{aligned}$$

9.6 We begin with the generating function for the set of all partitions, with respect to both size and length (= number of parts).

$$\sum_{\lambda \in \mathcal{Y}} x^{n(\lambda)} y^{k(\lambda)} = \prod_{j=1}^{\infty}\frac{1}{1 - x^j y}.$$

Setting $y = -1$ in this, the partition λ will contribute $x^{n(\lambda)}$ if $k(\lambda)$ is even, or $-x^{n(\lambda)}$ if $k(\lambda)$ is odd. Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty}(\mathrm{pe}(n) - \mathrm{po}(n))x^n &= \prod_{j=1}^{\infty}\frac{1}{1+x^j} \\
&= \prod_{j=1}^{\infty}\left(\frac{1}{1+x^j}\cdot\frac{1-x^j}{1-x^j}\right) \\
&= \prod_{j=1}^{\infty}\frac{1-x^j}{1-x^{2j}} \\
&= \prod_{j=1}^{\infty}(1-x^{2j-1}) \\
&= \sum_{\lambda\in\mathcal{OD}} x^{n(\lambda)}(-1)^{k(\lambda)} \\
&= \sum_{n=0}^{\infty}(-1)^n\mathrm{od}(n)x^n.
\end{aligned}$$

In this calculation $\mathcal{OD}$ is the set of partitions with odd and distinct parts, and the last equality uses the fact that, since $\lambda \in \mathcal{OD}$ is a partition in which every part is odd, it follows that $k(\lambda) \equiv n(\lambda) \pmod 2$. By equating the coefficients of x^n on both ends of this calculation we obtain the desired result.

9.8 (a)

$$\begin{aligned}
A(x,y) &= \sum_{\lambda\in\mathcal{Y}} x^{n(\lambda)}y^{m_1(\lambda)} \\
&= (1 + xy + x^2y^2 + x^3y^3 + \cdots)\prod_{j=2}^{\infty}\frac{1}{1-x^j} \\
&= \frac{1}{1-xy}\prod_{j=2}^{\infty}\frac{1}{1-x^j}
\end{aligned}$$

(b)

$$\begin{aligned} B(x,y) &= \sum_{\lambda\in\mathcal{Y}} x^{n(\lambda)} y^{b(\lambda)} \\ &= \prod_{j=1}^{\infty} \left(1 + x^j y + x^{2j} y + x^{3j} y + \cdots\right) \\ &= \prod_{j=1}^{\infty} \left(1 + \frac{x^j y}{1-x^j}\right) \end{aligned}$$

(c) To prove the result it suffices to show that

$$\frac{\partial}{\partial y} A(x,y)\bigg|_{y=1} = \frac{\partial}{\partial y} B(x,y)\bigg|_{y=1}.$$

By equating the coefficient of x^n on both sides, the result follows. Now

$$\begin{aligned} \frac{\partial}{\partial y} A(x,y)\bigg|_{y=1} &= \frac{x}{(1-x)^2} \prod_{j=2}^{\infty} \frac{1}{1-x^j} \\ &= \frac{x}{1-x} \prod_{j=1}^{\infty} \frac{1}{1-x^j} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial y} B(x,y)\bigg|_{y=1} &= \sum_{j=1}^{\infty} \frac{x^j}{1-x^j} \prod_{i\neq j}^{\infty} \frac{1}{1-x^i} \\ &= \left(\prod_{i=1}^{\infty} \frac{1}{1-x^i}\right) \sum_{j=1}^{\infty} x^j \\ &= \frac{x}{1-x} \prod_{i=1}^{\infty} \frac{1}{1-x^i}. \end{aligned}$$

That completes the proof.

10.3(a) The generating function for partitions is

$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{j=1}^{\infty} \frac{1}{1-x^j},$$

and by Euler's Pentagonal Number Theorem,

$$\prod_{j=1}^{\infty} (1-x^j) = \sum_{h=-\infty}^{\infty} (-1)^h x^{h(3h-1)/2}.$$

Therefore

$$\begin{aligned} 1 &= \prod_{j=1}^{\infty} \frac{1-x^j}{1-x^j} \\ &= \left(\sum_{i=0}^{\infty} p(i)x^n\right)\left(\sum_{h=-\infty}^{\infty} (-1)^h x^{h(3h-1)/2}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{h=-\infty}^{\infty} (-1)^h p(n-h(3h-1)/2)\right)x^n \end{aligned}$$

in which $p(i) = 0$ for all $i < 0$. Equating the coefficients of x^n on both sides yields

$$\sum_{h=-\infty}^{\infty} (-1)^h p\left(n - \frac{h(3h-1)}{2}\right) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

as claimed.

10.4 Recall the q-Binomial Theorem:

$$\prod_{i=1}^{n}(1+q^i x) = \sum_{k=0}^{n} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k.$$

Thus

$$\begin{aligned} &\lim_{n\to\infty} \sum_{k=0}^{n} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \\ = &\lim_{n\to\infty} \prod_{i=1}^{n}(1-q^i) \\ = &\prod_{i=1}^{\infty}(1-q^i) \\ = &\sum_{h=-\infty}^{\infty} (-1)^h q^{h(3h-1)/2}. \end{aligned}$$

The infinite product converges as a sequence of formal power series in $\mathbb{Z}[[q]]$ by Proposition 7.14.

10.11 To prove Gauss's identity

$$\prod_{j=1}^{\infty}(1+x^j)(1-x^{2j}) = \sum_{h=0}^{\infty} x^{h(h+1)/2}$$

we specialize the Jacobi Triple Product Formula

$$\sum_{h=-\infty}^{\infty} x^{h^2}y^h = \prod_{i=1}^{\infty}(1+x^{2i-1}y)(1+x^{2i-1}y^{-1})(1-x^{2i}).$$

Consider putting $x = y = t^{1/2}$. Then

$$x^{h^2}y^h = t^{h^2/2+h/2} = t^{h(h+1)/2},$$

which is good. Also notice that if $h \geq 0$ and $k = -1-h$ then

$$\frac{k(k+1)}{2} = \frac{(-1-h)(-1-h+1)}{2} = \frac{h(h+1)}{2}.$$

Therefore

$$\sum_{h=-\infty}^{\infty} t^{h(h+1)/2} = 2\sum_{h=0}^{\infty} t^{h(h+1)/2}.$$

Finally, the RHS of the JTPF specializes to

$$\begin{aligned}
& \prod_{i=1}^{\infty}(1+t^{(2i-1)/2}t^{1/2})(1+t^{(2i-1)/2}t^{-1/2})(1-t^{(2i)/2}) \\
= & \prod_{i=1}^{\infty}(1+t^i)(1+t^{i-1})(1-t^i) \\
= & \ 2\prod_{i=1}^{\infty}(1+t^i)^2(1-t^i) \\
= & \ 2\prod_{i=1}^{\infty}(1+t^i)(1-t^{2i}).
\end{aligned}$$

Comparing these results proves Gauss's identity.
