CO 330 Fall 2011 Solutions #6

due Friday, Dec.2.

Exercises: 11.5, 11.7, 11.8, 11.9, 11.12, 11.14(a), 11.15(a).

11.5.
(a) Let $\mathcal{D}$ be the class of derangements – that is, permutations with no fixed points. Let $\mathcal{E}$ be the class of sets, and for integer $k \geq 1$ let $\mathcal{C}_k$ be the class of k-cycles – that is, cyclic permutations on a set of k elements. Then $\mathcal{C}_{\geq 2} = \bigoplus_{k=2}^{\infty} \mathcal{C}_k$ is the class of cyclic permutations with at least 2 elements, and $\mathcal{D} \equiv \mathcal{E}[\mathcal{C}_{\geq 2}]$. Now, the exponential generating function of $\mathcal{D}$ is

$$\begin{aligned} D(x) &= \exp\left(\sum_{k=2}^{\infty} \frac{x^k}{k}\right) \\ &= \exp\left(\log\left(\frac{1}{1-x}\right) - x\right) \\ &= \frac{\exp(-x)}{1-x}, \end{aligned}$$

as claimed.

(b) From part (a) we calculate that

$$\begin{aligned} |\mathcal{D}_n| &= n![x^n]\frac{\exp(-x)}{1-x} \\ &= n![x^n]\left(\sum_{i=0}^{\infty}(-1)^i\frac{x^i}{i!}\right)\left(\sum_{j=0}^{\infty} x^j\right) \\ &= n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}, \end{aligned}$$

as was to be shown.

11.7. By Example 11.19, the class of set partitions is $\mathcal{E}[\mathcal{E}_{\geq 1}]$. For $0 \leq k \leq n$, let $S(n,k)$ be the number of partitions of the set $\{1, 2, ..., n\}$

with k blocks. Then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} S(n,k)y^k \frac{x^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{\pi\in Ptn_n} y^{|\pi|}\right)\frac{x^n}{n!} = \exp(y\exp(x)-y).$$

Therefore, for $0 \le k \le n$,

$$\begin{aligned} S(n,k) &= n![x^n y^k]\exp(y\exp(x)-y) \\ &= n![x^n y^k]\sum_{j=0}^{\infty}\frac{y^j(\exp(x)-1)^j}{j!} \\ &= \frac{n!}{k!}[x^n]\sum_{i=0}^{k}\binom{k}{i}\exp(x)^i(-1)^{n-i} \\ &= \frac{n!}{k!}[x^n]\sum_{i=0}^{k}(-1)^{n-i}\binom{k}{i}\frac{i^n}{n!} \\ &= \frac{1}{k!}\sum_{i=0}^{k}(-1)^{n-i}\binom{k}{i}i^n, \end{aligned}$$

as in Exercise 3.11.

11.8. Let $\mathcal{S}$ be the class of permutations, $\mathcal{C}$ the class of cyclic permutations, and $\mathcal{E}$ the class of sets. Let $c(\sigma, k)$ be the number of cycles of length k in the permutation σ, and consider

$$S(x,y) = \sum_{n=0}^{\infty}\left(\sum_{\sigma\in\mathcal{S}_n} y^{c(\sigma,k)}\right)\frac{x^n}{n!}.$$

(a) Recall that $\mathcal{S} \equiv \mathcal{E}[\mathcal{C}]$, and let $\mathcal{C}_k$ be the class of k-cycles. Thus $S(x,y) = \exp(C(x,y))$ in which

$$C(x,y) = \sum_{j=1}^{\infty} y^{\chi[j=k]}\frac{x^j}{j} = \log\left(\frac{1}{1-x}\right) + \frac{x^k}{k}(y-1).$$

(Here, as usual, for a proposition $\mathtt{P}$ we have $\chi[\mathtt{P}] = 1$ if $\mathtt{P}$ is true, and $\chi[\mathtt{P}] = 0$ if $\mathtt{P}$ is false.) That is,

$$S(x,y) = \exp\left(\log\left(\frac{1}{1-x}\right) + \frac{x^k}{k}(y-1)\right) = \frac{\exp(x^k(y-1)/k)}{1-x}.$$

(b) From part (a) the average number of k-cycles among all $n!$ permutations of the set $\{1, 2, ..., n\}$ is

$$\begin{aligned} & \frac{1}{n!} n! [x^n] \left. \frac{\partial}{\partial y} \frac{\exp(x^k (y-1)/k)}{1-x} \right|_{y=1} \\ = \; & [x^n] \left. \frac{1}{1-x} \exp(x^k (y-1)/k) x^k / k \right|_{y=1} \\ = \; & \frac{1}{k} [x^n] \frac{x^k}{1-x} \\ = \; & \begin{cases} 1/k & \text{if } 1 \le k \le n, \\ 0 & \text{if } n < k, \end{cases} \end{aligned}$$

as was to be shown.

11.9. Let $\mathcal{Y}$ be the class of labelled (unrooted) trees in which each vertex has degree 1 or 3. Let $\mathcal{Q}$ be the class of labelled *rooted* trees in which each vertex has either 0 or 2 children. For each $k \in \mathbb{N}$, let $\mathcal{E}_k$ be the class of k-sets. Then

$$\mathcal{Y}^{\bullet} \equiv \mathcal{E}_1 * (\mathcal{Q} \oplus \mathcal{E}_3[\mathcal{Q}])$$

and

$$\mathcal{Q} \equiv \mathcal{E}_1 * (\mathcal{E}_0 \oplus \mathcal{E}_2[\mathcal{Q}]),$$

as is easily seen by drawing a suitably general picture of a tree in $\mathcal{Y}$. The exponential generating functions of these classes satisfy the equations

$$Y^{\bullet} = x(Q + Q^3/6)$$

and

$$Q = x(1 + Q^2/2),$$

respectively. Therefore (applying LIFT) we see that for $n \ge 2$, the number of trees in $\mathcal{Y}$ on the vertex-set $\{1, 2, ..., n\}$ is

$$\begin{aligned} |\mathcal{Y}_n| &= \frac{1}{n} |\mathcal{Y}_n^{\bullet}| = \frac{1}{n} n! [x^n] Y^{\bullet}(x) \\ &= (n-1)! [x^n] x(Q + Q^3/6). \end{aligned}$$

We use LIFT with $F(u) = u + u^3/6$ and $G(u) = 1 + u^2/2$. Notice that $F'(u) = 1 + u^2/2 = G(u)$, which is nice. Applying LIFT, we have

$$\begin{aligned} |\mathcal{Y}_n| &= (n-1)![x^{n-1}]F(Q) \\ &= (n-2)![u^{n-2}](1+u^2/2)^n \\ &= (n-2)![u^{n-2}]\sum_{k=0}^{n}\binom{n}{k}\frac{u^{2k}}{2^k} \\ &= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{(2k)!}{2^k}\binom{2k+2}{k} & \text{if } n = 2k+2. \end{cases} \end{aligned}$$

This is what was to be shown.

11.12. Let $\mathcal{K}$ be the class of triangle-trees, and consider a rooted triangle-tree $(T, v) \in \mathcal{K}^\bullet_X$ for some finite set X. Deleting v, $T \setminus v$ decomposes naturally as a set of connected components, each of which is naturally an unordered pair of rooted triangle-trees in $\mathcal{K}^\bullet$. That is,

$$\mathcal{K}^\bullet \equiv \mathcal{E}_1 * \mathcal{E}[\mathcal{E}_2[\mathcal{K}^\bullet]].$$

So for the exponential generating functions we have the equation

$$K^\bullet = x\,(\exp((K^\bullet)^2/2).$$

A straightforward application of LIFT completes the calculation:

$$\begin{aligned} |\mathcal{K}_n| &= \frac{1}{n}|\mathcal{K}^\bullet_n| = (n-1)![x^n]K^\bullet(x) \\ &= \frac{(n-1)!}{n}[u^{n-1}]\exp(u^2/2)^n \\ &= \frac{(n-1)!}{n}[u^{n-1}]\sum_{j=0}^{\infty}\frac{n^j u^{2j}}{j!2^j} \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{(2j)!(2j+1)^{j-1}}{j!2^j} & \text{if } n = 2j+1 \geq 1. \end{cases} \end{aligned}$$

There it is!

11.14(a). Let $\mathcal{Q}$ be the class of oriented cacti, and consider a rooted oriented cactus $(Q, v) \in \mathcal{Q}^\bullet_X$ for some finite set X. Deleting the root vertex $v \in X$ from Q we obtain $Q \setminus v$. Every connected component of $Q \setminus v$ is a nonempty totally ordered sequence of structures in $\mathcal{Q}^\bullet$. That is,

$$\mathcal{Q}^\bullet \equiv \mathcal{E}_1 * \mathcal{E}[\mathcal{Q}^\bullet * (\mathcal{Q}^\bullet)^*].$$

Hence, for the exponential generating function we have the equation

$$Q^\bullet = x\,\exp(Q^\bullet/(1-Q^\bullet)).$$

Now we apply LIFT:

$$\begin{aligned}
|\mathcal{Q}_n| &= \frac{1}{n}|\mathcal{Q}^\bullet_n| = (n-1)![x^n]Q^\bullet(x)\\
&= \frac{(n-1)!}{n}[u^{n-1}]\exp(u/(1-u))^n\\
&= \frac{(n-1)!}{n}[u^{n-1}]\sum_{j=0}^{\infty}\frac{n^j u^j}{j!(1-u)^j}\\
&= \frac{(n-1)!}{n}\sum_{j=0}^{n-1}\frac{n^j}{j!}[u^{n-1-j}]\sum_{i=0}^{\infty}\binom{i+j-1}{j-1}u^i\\
&= (n-1)!\sum_{j=0}^{n-1}\frac{n^{j-1}}{j!}\binom{(n-1-j)+j-1}{j-1}\\
&= (n-1)!\sum_{j=0}^{n-1}\frac{n^{j-1}}{j!}\binom{n-2}{j-1}\\
&= (n-1)!\sum_{j=0}^{n-1}\frac{n^{j-1}}{j!}\binom{n-2}{n-1-j}.
\end{aligned}$$

Done! ;-D

11.15(a). Let $\mathcal{T}$ be the class of trees, and let $\mathcal{R} = \mathcal{T}^\bullet$ be the class of rooted trees. Then $\mathcal{R} \equiv \mathcal{E}_1 * \mathcal{E}[\mathcal{R}]$ as we have seen many times. Let $\tau(T,v)$ be the number of terminal vertices of the rooted tree (T,v), and consider the mixed bivariate generating function

$$R(x,y) = \sum_{n=0}^{\infty}\left(\sum_{(T,v)\in\mathcal{R}_n} y^{\tau(T,v)}\right)\frac{x^n}{n!}$$

Consider this equivalence of classes:

$$\begin{aligned}
\mathcal{R} &\equiv \mathcal{E}_1 * \mathcal{E}[\mathcal{R}]\\
\mathcal{R}_X &\rightleftharpoons (\mathcal{E}_1 * \mathcal{E}[R])_X\\
(T,v) &\leftrightarrow (v, \{(S_1,w_1),...,(S_k,w_k)\})\\
n(T) &= 1 + n(S_1) + \cdots + n(S_k)\\
\tau(T,v) &= \chi[k=0] + \tau(S_1,w_1) + \cdots + \tau(S_k,w_k).
\end{aligned}$$

We see that the exponential generating function $R(x,y)$ satisfies the equation

$$R = x(\exp(R) + y - 1).$$

Sweet! So the number of rooted trees on the vertex-set $\{1, 2..., n\}$ that have exactly k terminal vertices is (by LIFT)

$$\begin{aligned} & n![x^n y^k]R(x,y) \\ = \; & (n-1)![y^k u^{n-1}](\exp(u) + y - 1)^n \\ = \; & (n-1)![y^k u^{n-1}] \sum_{i=0}^{n} \binom{n}{i} y^i (\exp(u) - 1)^{n-i} \\ = \; & (n-1)![u^{n-1}] \binom{n}{k} (\exp(u) - 1)^{n-k}. \end{aligned}$$

To compare, since the class of set partitions is $\mathcal{E}[\mathcal{E}_{\geq 1}]$, for integers $0 \leq b \leq a$,

$$S(a,b) = a![q^b t^a] \exp(q \exp(t) - qt) = \frac{a!}{b!}[t^a](\exp(t) - 1)^b$$

So

$$S(n-1, n-k) = \frac{(n-1)!}{(n-k)!}[t^{n-1}](\exp(t) - 1)^{n-k}.$$

The same with u in place of t, from which we continue the thread of the main computation:

$$\begin{aligned} & n![x^n y^k]R(x,y) \\ = \; & (n-1)![u^{n-1}] \frac{n!}{k!(n-k)!} (\exp(u) - 1)^{n-k} \\ = \; & \frac{n!}{k!} \cdot \frac{(n-1)!}{(n-k)!}[u^{n-1}](\exp(u) - 1)^{n-k}. \\ = \; & (n-k)! \binom{n}{k} \cdot S(n-1, n-k). \end{aligned}$$

That's what we wanted. Yay! ;-p
