C&O 331: Midterm Test

Duration: 2 hours   Aids: None   Total marks: 43

NAME and ID: _____

1. [6 marks] Let C be an (n, k)-code over $F = GF(q)$.

   (a) Define the *dual code* $C^\perp$ of C. [2 marks]

   (b) Give the definition of a *parity-check matrix* for C. [2 marks]

   (c) Can C have more than one parity-check matrix? (Explain) [2 marks]

2. [10 marks] Let $\mathbb{Z}_7$ be the finite field with 7 elements.

   (a) Prove that the polynomial $x^2 + 4$ is irreducible in $\mathbb{Z}_7[x]$. [2 marks]

   (b) How many elements does the field $F = \mathbb{Z}_7[x]/(x^2 + 4)$ have? [2 marks]

   (c) Write down eight distinct elements of F. [2 marks]

   (d) Calculate $(x + 1)^7$ in F. [2 marks]

   (e) Does F have an element of order 7? Does F have an element of order 8? (Explain) [2 marks]

3. [5 marks] Let C be an $[n, M]$-code of distance d over $F = GF(q)$, with $e = \lfloor \frac{d-1}{2} \rfloor$.

(a) Define what it means for C to be a *perfect code*. [2 marks]

(b) Prove that the n-fold repetition code $C = \{\underbrace{000\ldots0}_{n \text{ times}}, \underbrace{111\ldots1}_{n \text{ times}}\} \subset V_n(\mathbb{Z}_2)$ is a perfect binary code with distance $d = n$ if d is odd. [3 marks]

4. [8 marks] Consider the single-error correcting linear (n, k) code C over $\mathbb{Z}_3$ with parity-check matrix

$$H = \begin{bmatrix} 2 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Determine n and k of C. [2 marks]

(b) Determine the distance d of C. [3 marks]

(c) Correct the received vector $r = (10220011)$ to the nearest codeword, if possible. [3 marks]

5. (4 marks) Let C be an (n, k)-code over F with distance d, and let $e = \lfloor (d-1)/2 \rfloor$. Let $x \in V_n(F)$ be a vector of weight $w(x) \leq e$. Prove that x is the unique vector of minimum weight in its coset of C.

6. [10 marks] Let C be the linear code over $GF(3)$ with parity-check matrix

$$
H = \begin{bmatrix}
2 & 0 & 2 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
2 & 1 & 2 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\
2 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\
2 & 2 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

The following is a partial list of coset leaders and their corresponding syndromes under the matrix H.

Coset Leader	Syndrome
00000000000	00000
10000000000	22222
01000000000	?
00100000000	20211
00010000000	?
00001000000	11202
00000100000	21120
00000010000	10000
00000001000	01000
00000000100	00100
00000000010	00010
00000000001	00001

(a) Fill in the missing entries in the table. [2 marks]

(b) Suppose the vector $r = (01220000100)$ is received. Correct r to the nearest codeword. [4 marks]

(c) Suppose the vector $r = (00012100000)$ is received. Correct r to the nearest codeword. [4 marks]

EXTRA PAGE-1.

EXTRA PAGE-2.