CO342 ASSIGNMENT #10
DUE: IN CLASS WEDNESDAY 20 JULY 2011

1. (a) Let G be a connected graph and let A be any non-empty subset of $V(G)$. Prove that G contains a (not necessarily spanning) tree T so that every leaf of T **(typo corrected)** is in A and every vertex of A is in T (not necessarily as a leaf).
   (b) Let C be a cycle in a connected graph G and let B be a C-bridge. Let A denote the set of attachments of B (these are the vertices in $B \cap C$). Prove that B contains a tree T so that the leaves of T are precisely the vertices in A.

2. [**15 points**] Let $m \geq 3$ and $n \geq 3$ be positive integers and let $\Pi_{m,n}$ denote the graph with vertex set consisting of all the ordered pairs (i, j), with i in $\{0, 1, 2, \ldots, m-1\}$ and j in $\{0, 1, 2, \ldots, n-1\}$, and the vertex (i, j) is adjacent to the four vertices $(i+1, j)$, $(i-1, j)$, $(i, j+1)$ and $(i, j-1)$. We remark that first coordinates are numbers read modulo m, while second coordinates are numbers read modulo n. In particular, $(0, n-1)$ is adjacent to $(1, n-1)$, $(m-1, n-1)$, $(0, 0)$, and $(0, n-2)$. See the figure for two different drawings of the graph $\Pi_{4,6}$.

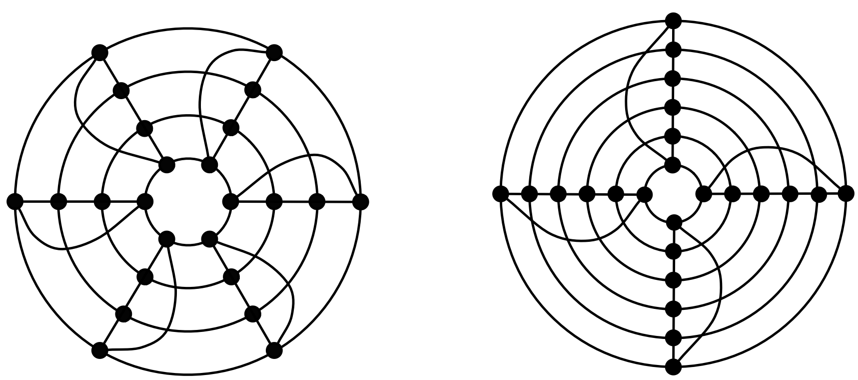

Figure 1: Two drawings of $\Pi_{4,6}$.

Here are some problems about $\Pi_{m,n}$ ([**5 points each**]).

(a) Prove that, for each $i = 0, 1, \ldots, m-1$ and $j = 0, 1, 2, \ldots, n-1$, the 4-cycle

$$((i,j),(i+1,j),(i+1,j+1),(i,j+1),(i,j))$$

is a peripheral cycle, where the first coordinates are read modulo m and the second are read modulo n.

(b) Prove that, for each $i = 0, 1, 2 \ldots, m-1$, the n-cycle C_i defined by

$$C_i = ((i,0),(i,1),(i,2),\ldots,(i,n-1),(i,0))$$

is peripheral.

(c) Prove that, for $i = 1, 2, \ldots, m-1$, $E(C_i)$ is a linear combination of $E(C_0)$ and the edge sets of the cycles mentioned in (2a).
Alternatively, for 10 extra bonus points = 15 points total prove that the edge sets of the cycles mentioned in (2a), $E(C_0)$, and the edge set of the cycle

$$((0,0),(1,0),(2,0),\ldots,(m-1,0),(0,0))$$

(typo corrected) span the cycle space of G.

3. Let C be a cycle in a graph G. The *overlap diagram* $\mathrm{OD}(C)$ is a new graph having a vertex for each C-bridge, and an edge joining any two vertices corresponding to overlapping C-bridges.

   Prove: if G is planar, then $\mathrm{OD}(G)$ is bipartite. *(Hint: relate the bipartition to how the C-bridges sit in a planar embedding of G.)*

4. Let G be a connected graph and let H and K be subgraphs of G so that $G = H \cup K$. Suppose G contains a subdivision of $K_{3,3}$.

   (a) Suppose $H \cap K$ is just one vertex. Show that either H or K contains a subdivision of $K_{3,3}$.

   (b) Suppose $H \cap K$ is just two vertices u and v so that both $H - \{u,v\}$ and $K - \{u,v\}$ are connected. Show that either $H + uv$ or $K + uv$ contains a subdivision of $K_{3,3}$.
   (Hint: If both $H - \{u,v\}$ and $K - \{u,v\}$ had one of the degree-3 vertices of the $K_{3,3}$-subdivision L, then they are joined by 3 internally-disjoint paths in L, which is impossible in G.)