CO342 ASSIGNMENT #10
DUE: IN CLASS WEDNESDAY 20 JULY 2011

1. (a) Let G be a connected graph and let A be any non-empty subset of $V(G)$. Prove that G contains a (not necessarily spanning) tree T so that every leaf of T **(typo corrected)** is in A and every vertex of A is in T (not necessarily as a leaf).

   SOLUTION. *There are two kinds of solutions. In the first, we let $a_1, a_2, \dots, a_k$ be the vertices in A and one at at time add them to the tree. Let T_1 be the first tree, consisting just of a_1. Given tree T_i that contains at least $a_1, a_2, \dots, a_i$, and every leaf of T_i is in A, then either every vertex of A is already in T_i and we are done, or there is a least $j > i$ so that a_j is not yet in T_i. In that case, we let P be a shortest path joining a_j to a vertex in T_i; thus, P has a_j as one end and the intersection of P with T_i is the other end of P. Now set $T_{i+1} = T_i \cup P$. It is easy to see that any leaf of T_{i+1} is either a leaf of T_i or a_j. Eventually there is a least i so that T_i contains all the vertices in A; all the leaves of T_i are in A.*

   In the other solution, let T be a minimal tree in T containing all the vertices in A. (Any spanning tree of G will have this property, but it might have some leaves that are not in A.) If some leaf v of T is not in A, then $T - v$ is a proper subtree of T that contains all the vertices in A, contradicting the minimality of T. Therefore, all the leaves of T are in A.

   (b) Let C be a cycle in a connected graph G and let B be a C-bridge. Let A denote the set of attachments of B (these are the vertices in $B \cap C$). Prove that B contains a tree T so that the leaves of T are precisely the vertices in A.

   SOLUTION. *If B is just an edge whose ends are both in C, then B itself is such a tree. Thus, we may assume B arises from a component K of $G - V(C)$ by the addition of edges with one end in K and one end in C, together with the ends in C of these edges.*

   For each vertex v of $B \cap C$, there is a vertex w_v of K adjacent to v. Let A consist of the set of these w_v. Since K is connected, *Part (a) implies there is a tree T in K containing all the w_v so that every leaf of T is in A. Now add the vertices in $B \cap C$ to T, along with the edges vw_v; this yields the desired tree.*

2. [**15 points**] Let $m \geq 3$ and $n \geq 3$ be positive integers and let $\Pi_{m,n}$ denote the graph with vertex set consisting of all the ordered pairs (i, j), with i in $\{0, 1, 2, \ldots, m-1\}$ and j in $\{0, 1, 2, \ldots, n-1\}$, and the vertex (i, j) is adjacent to the four vertices $(i+1, j)$, $(i-1, j)$, $(i, j+1)$ and $(i, j-1)$. We remark that first coordinates are numbers read modulo m, while second coordinates are numbers read modulo n. In particular, $(0, n-1)$ is adjacent to $(1, n-1)$, $(m-1, n-1)$, $(0, 0)$, and $(0, n-2)$. See the figure for two different drawings of the graph $\Pi_{4,6}$.

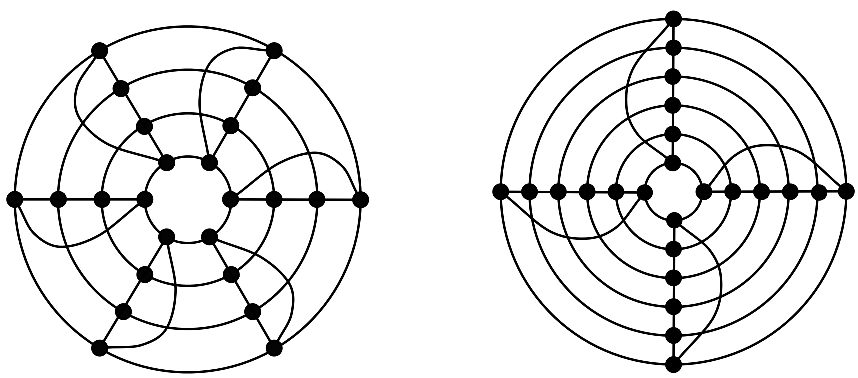

Figure 1: Two drawings of $\Pi_{4,6}$.

Here are some problems about $\Pi_{m,n}$ ([**5 points each**]).

(a) Prove that, for each $i = 0, 1, \ldots, m-1$ and $j = 0, 1, 2, \ldots, n-1$, the 4-cycle

$$((i, j), (i+1, j), (i+1, j+1), (i, j+1), (i, j))$$

is a peripheral cycle, where the first coordinates are read modulo m and the second are read modulo n.

SOLUTION. *Neither of the adjacencies $(i, j)(i+1, j+1)$ nor $(i+1, j)(i, j+1)$ occurs in G, so, for the 4-cycle*

$$C = ((i, j), (i+1, j), (i+1, j+1), (i, j+1), (i, j)),$$

there is no C-bridge consisting of an edge not in C joining two vertices in C.

Since $m \geq 3$ and $n \geq 3$, there exist $k \neq i, i+1$ so that $0 \leq k < m$, and $\ell \neq j, j+1$ so that $0 \leq \ell < n$. For each vertex (r,s) of G not in C, there is a path in the cycle $((r,s),(r,s+1),(r,s+2),\ldots,(r,s-1))$ (second coordinates read modulo n) that does not go through (r,j) and $(r,j+1)$, but joins (r,s) to (r,ℓ). We may then join (r,ℓ) to (k,ℓ) using the cycle $((0,\ell),(1,\ell),\ldots,(m-1,\ell))$ — this is disjoint from C. Thus, every vertex of $G-V(C)$ is joined by a path in $G-V(C)$ to (k,ℓ) and, therefore, $G-V(C)$ is connected.

We conclude that there is only one C-bridge, as required.

(b) Prove that, for each $i = 0, 1, 2 \ldots, m-1$, the n-cycle C_i defined by

$$C_i = ((i,0),(i,1),(i,2),\ldots,(i,n-1),(i,0))$$

is peripheral.

SOLUTION. *The only edges of the form $(i,j)(i,k)$ have $k \equiv j \pm 1 \pmod{n}$. Therefore, there is no edge of C_i joining two vertices of C_i except for the edges of C_i.*

Every vertex (j,k) in $G-V(C_i)$ has first coordinate different from i. We can join (j,k) to $(i-1,0)$, first by a $(j,k)(i-1,k)$-path in the cycle $((0,k),(1,k),\ldots,(m-1,k),(0,k))$ that avoids (i,k), and then by an $(i-1,k)(i-1,0)$-path in the cycle C_{i-1}. Therefore, $G-V(C_i)$ is connected, as required.

(c) Prove that, for $i = 1, 2, \ldots, m-1$, $E(C_i)$ is a linear combination of $E(C_0)$ and the edge sets of the cycles mentioned in (2a).

SOLUTION. *Let $Q_{i,j}$ be the cycle*

$$((i,j),(i+1,j),(i+1,j+1),(i,j+1),(i,j))\,.$$

It is easy to check that $E(C_i) = E(C_{i-1}) \oplus X_i$, where

$$X_i = \bigoplus_{j=0}^{n-1} E(Q_{i-1,j})\,.$$

Now a simple induction on i shows that $E(C_i)$ is a linear combination of $E(C_0)$ and the $E(Q_{k,\ell})$.

Alternatively, for 10 extra bonus points = 15 points total prove that the edge sets of the cycles mentioned in (2a), $E(C_0)$, and the edge set of the cycle

$$((0,0),(1,0),(2,0),\dots,(m-1,0),(0,0))$$

(typo corrected) span the cycle space of $\Pi_{m,n}$ **(typo corrected)**.

SOLUTION. *Let z be in the cycle space of $\Pi_{m,n}$. We can use the squares $((j,k),(j+1,k),(j+1,k+1),(j,k+1))$ to eliminate all edges in z that join any (i,k) to $(i,k+1)$, except at most one, which we can arrange (with these squares) to be $(i,0)(i+1,0)$. After all this elimination, we have an element z' of the cycle space of $\Pi_{m,n}$ whose edges are in C_0, C_1, ..., C_{m-1}, and the cycle $((0,0),(1,0),(2,0),\dots,(m-1,0),(0,0))$.*

If some edge of any C_i is in z', then $E(C_i)\subseteq z'$, as otherwise there is a vertex of $C_i-(i,0)$ that is incident with only one edge of z', a contradiction. From the easier version of this question, each C_i is a sum of C_0 and squares. Therefore, we may eliminate all the C_i from z' to get an element z'' of the cycle space that is contained in

$$E(\,((0,0),(1,0),(2,0),\dots,(m-1,0),(0,0))\,)\,,.$$

Either z'' is all the edges in this cycle or it is empty; in either case, we have z is a linear combination of C_0, $((0,0),(1,0),(2,0),\dots,$ $(m-1,0),(0,0))$, and the squares, as required.

3. Let C be a cycle in a graph G. The *overlap diagram* $\mathrm{OD}(C)$ is a new graph having a vertex for each C-bridge, and an edge joining any two vertices corresponding to overlapping C-bridges.

   Prove: if G is planar, then $\mathrm{OD}(G)$ is bipartite. *(Hint: relate the bipartition to how the C-bridges sit in a planar embedding of G.)*

   SOLUTION. *Suppose two C-bridges B_1 and B_2 are on the same side of C in a planar embedding of G. We claim that B_1 and B_2 avoid each other. Otherwise, they overlap, and there are two cases to consider.*

   If there are attachments x_1 and y_1 of B_1 and x_2 and y_2 of B_2 so that x_1,x_2,y_1,y_2 are distinct and occur in this cyclic order on C, then, for

$i = 1, 2$, let P_i be an x_iy_i-path in B_i so that $P_i \cap C$ is just x_i and y_i. Then P_1 and P_2 are totally disjoint.

The disjoint paths P_1 and P_2 are on the same side of C in the embedding, and therefore they must cross, which is impossible.

In the other case, B_1 and B_2 both have precisely three attachments, and they are the same three vertices x, y, z. For $i = 1, 2$. Question 1 (b) implies there is a tree T_i in B_i so that the leaves of T_i are precisely x, y, z. Then T_i has exactly one degree 3 vertex v_i, which is joined to x, y, z in T_i by paths P^i_x, P^i_y, and P^i_z, respectively.

In the planar embedding of G, T_1 is embedded on one side of C and v_2 is embedded on that same side of C. Thus, v_2 is in one of the faces of $C \cup T_1$; we may assume this is the one bounded by $P^1_x \cup P^1_y$ and the xy-subpath of C that does not contain z. But now the path P^2_z must cross this boundary cycle, again a contradiction.

This shows that, indeed, B_1 and B_2 avoid each other. This implies that two C-bridges on the same side of C in the embedding are not adjacent in OD(G). Thus, every edge of OD(G) joins C-bridges that are on different sides of C in the planar embedding of G. Since C has only two sides, this implies that OD(G) is bipartite, as claimed.

4. Let G be a connected graph and let H and K be subgraphs of G so that $G = H \cup K$. Suppose G contains a subdivision of $K_{3,3}$.

   (a) Suppose $H \cap K$ is just one vertex. Show that either H or K contains a subdivision of $K_{3,3}$.

   SOLUTION. *Let v be the vertex common to H and K and let L be a subdivision of $K_{3,3}$ in G. We claim that either $L \subseteq H$ or $L \subseteq K$. The alternative is that some vertex v_H of L is in $H - V(K)$ and some other vertex v_L of L is in $K - V(H)$. Since L is a 2-connected graph, there is a cycle C of L containing both v_H and v_L.*

   On the other hand, $C - v$ gives a path in $G - v$ joining v_H and v_L, which is impossible. Therefore, either L has no vertices in $H - V(K)$ or L has no vertices in $K - V(H)$. That is, either $L \subseteq K$ or $L \subseteq H$.

(b) Suppose $H \cap K$ is just two vertices u and v so that both $H - \{u, v\}$ and $K - \{u, v\}$ are connected. Show that either $H + uv$ or $K + uv$ contains a subdivision of $K_{3,3}$.

(Hint: If both $H - \{u, v\}$ and $K - \{u, v\}$ had one of the degree-3 vertices of the $K_{3,3}$-subdivision L, then they are joined by 3 internally-disjoint paths in L, which is impossible in G.)

SOLUTION. *Suppose $H - V(K)$ contains a vertex v_H of degree 3 in L and $K - V(H)$ contains a vertex v_K of degree 3 in L. In L there are three internally disjoint $v_H v_K$-paths. These are, of course, paths in G. But this contradicts Menger's Theorem and the fact that v_H and v_K are in different components of $G - \{u, v\}$. Therefore, all the degree 3 vertices of L are in the same one of H and K. We may choose the labelling so they are all in H.*

It may happen that some edge e of L is in K. But such an edge is in some path P of L joining two degree 3 vertices in L. In particular, P must contain both u and v, and we may replace the uv-subpath of P with the edge uv to see that this new subdivision of $K_{3,3}$ is contained in $H + uv$.