CO342 ASSIGNMENT #1

1. Let G be a graph, having at least two edges, with the following properties: (i) for every edge e of G, $G - e$ is connected; and (ii) for any distinct edges e, f of G, $(G - e) - f$ is not connected. Prove that G is a cycle.

   SOLUTION. *Let uv be an edge of G and let P be a uv-path in $G - e$. Then $P + uv$ is a cycle C in G.*

   Suppose f is an edge of G not in C. By (i), $G - f$ is connected and obviously contains C. But then the edge uv is in C in $G - f$ and so is not a bridge of $G - f$. Therefore, $(G - f) - uv$ is connected, contradicting (ii). It follows that every edge of G is in C.

   If v is a vertex of G not in C, then, as G is connected, v is incident with an edge of G that is not in C, contradicting the conclusion of the preceding paragraph. Thus, every vertex of G is also in C.

2. For every positive integer n with $n \geq 3$, give an example of a graph G on n vertices that has both an avoidable and an unavoidable vertex.

   SOLUTION. *If $m < n$, then the graph $K_{m,n}$ has the n vertices of degree m avoidable; while the m vertices of degree n are unavoidable.*

3. Let m and n be positive integers. Determine the avoidable and unavoidable vertices in:

   (a) the complete graph K_n;

   SOLUTION. *If n is even, then all vertices are unavoidable. If n is odd, then all vertices are avoidable.*

   (b) the complete bipartite graph $K_{m,n}$;

   SOLUTION. *When $m < n$, see Question 2. For $m = n$, all vertices are unavoidable.*

   (c) the n-dimensional cube Q_n.

   SOLUTION. *As we discussed in class, Q_n has a perfect matching, so all vertices are unavoidable.*

4. Let v be a vertex in the n-dimensional cube Q_n. Determine the avoidable and unavoidable vertices in $Q_n - v$.

   SOLUTION. *Since Q_n is bipartite and has a perfect matching, the two sides A and B of the bipartition of Q_n have the same size. Let A be the part of the bipartition of Q_n containing v. Then $\nu(Q_n - v) = \nu(Q_n) - 1$, so a maximum matching in $Q_n - v$ must saturate all the vertices of A — so these are unavoidable in $Q_n - v$ — and saturate all but one of the vertices in B.*

   We claim that all the vertices of B are avoidable. We proceed by induction, the base case $n = 1$ being trivial. We may assume $n \geq 2$ and $v = 000\ldots0$. If $u \in B$ is not $111\ldots1$, then there is a coordinate of u that is 0. All the vertices with that coordinate 0 yields a subgraph isomorphic to Q_{n-1} and the induction applied to that subgraph gives a matching M that saturates every vertex except u and v. A perfect matching in the subgraph with that coordinate always equal to 1 shows u is avoidable.

   In the remaining case, n is odd and $u = 111\ldots1$. In this case, consider the subgraph isomorphic to Q_{n-1} with all first coordinates equal to 0. There is a matching in this subgraph that saturates every vertex except v and $001\ldots1$. In the subgraph isomorphic to Q_{n-1} with all first coordinates equal to 1, there is a matching that saturates every vertex except u and $101\ldots1$. To these two matchings we add the edge of Q_n joining $001\ldots1$ and $101\ldots1$ to get the matching that misses just u and v.

5. Let T be a tree having at least two vertices. Let v be a vertex of T with degree 1 and let w be its neighbour in T.

   (a) Prove that $\nu(T) = 1 + \nu(T - \{v, w\})$.

   SOLUTION. *Since v and w are adjacent in T, $\nu(T - \{v, w\} \leq \nu(T) - 1$, since, for any matching M in $T - \{v, w\}$, $M \cup \{vw\}$ is a matching in T.*

   On the other hand, suppose M is a matching in T. Then either $vw \in M$ or v is unsaturated by M. In the former case, $M - \{vw\}$ is a matching in $T - \{v, w\}$ with size $|M| - 1$. In the latter case, M is a matching in $G - v$; therefore, there is a matching of size at *least* $|M|-1$ *in* $G-\{v,w\}$. *These remarks prove* $\nu(T-\{v,w\}) \geq \nu(T)-1$.

(b) Based on 5a, describe how to find a maximum matching in a tree. (Do not forget to take into account that $T-\{v,w\}$ might not be connected.)

SOLUTION. *Start with forest* $F_0 = T$ *and* $M_0 = \emptyset$. *As long as* F_i *has an edge, let* v_i *be a vertex of degree 1 in* F_i, w_i *its neighbour in* F_i, *set* $M_{i+1} = M_i \cup \{v_i w_i\}$ *and* $F_{i+1} = F_i - \{v_i, w_i\}$.

At termination, F_k *has no edges and* M_k *is a maximum matching in* T. *The proof that* M_k *is a maximum matching uses the fact that the maximum matching in a graph is composed of maximum matchings in each component. Thus, 5a implies that* $v_i w_i$ *is in some maximum matching of* F_i *and so* $\nu(F_{i+1}) = \nu(F_i) - 1$. *It is now a trivial induction to see that* $\nu(T) = \nu(F_i) + i$ *and, therefore,* $\nu(T) = k$ *(with* k *being the first index for which* F_k *has no edges.*

(c) Based on 5a, or otherwise, prove that a tree has at most one perfect matching.

SOLUTION. *Let* M *be a perfect matching in* T, *let* v *be a vertex of* T *with degree 1 in* T, *and let* w *be its neighbour in* T. *Then 5a implies* $vw \in M$, *so* $M-\{vw\}$ *is a perfect matching in* $T-\{v,w\}$. *By induction, each component of* $T-\{v,w\}$ *has at most one perfect matching, so* $M-\{vw\}$ *is uniquely determined. Therefore, so is* M.

The base of the induction is $T = K_2$, *which obviously has only one perfect matching.*