CO342 ASSIGNMENT #2
DUE: IN CLASS WEDNESDAY 18 MAY 2011

The *Maximum Matching Formula* is:

$$\nu(G) = \frac{1}{2} \min \big\{ |V(G)| - \mathrm{odd}(G - S) + |S| \ : \ S \subseteq V(G) \big\} .$$

The *Perfect Matching Criterion* is: G has a perfect matching if and only if, for every $S \subseteq V(G)$,

$$\mathrm{odd}(G - S) \leq |S| .$$

1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be sets of (finite) sets with the property that, for every $A \in \mathfrak{A}$ and every $B \in \mathfrak{B}$, $|A| \leq |B|$. Suppose that there is a set $A^* \in \mathfrak{A}$ and a set $B_* \in \mathfrak{B}$ so that $|A^*| = |B_*|$. Prove that

$$|A^*| = \max\{|A| : A \in \mathfrak{A}\} \text{ and } |B_*| = \min\{|B| : B \in \mathfrak{B}\} .$$

2. Let G be a graph and let $S \subseteq V(G)$. Recall $\mathrm{odd}(G - S)$ is the number of components of $G - S$ that have an odd number of vertices. Let M be an matching in G. Prove that

$$|M| \leq \frac{1}{2} \left(|V(G)| - \mathrm{odd}(G - S) + |S| \right) .$$

3. Let G be the graph in the figure and let T consist of the three red vertices.

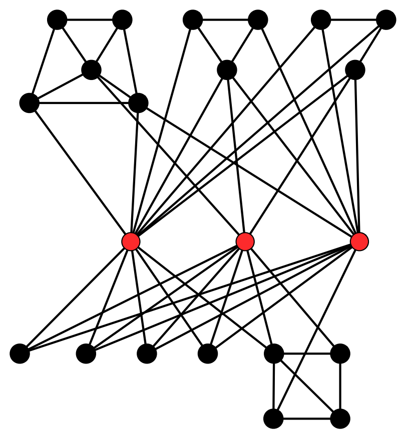
   (a) Find a matching M in G so that

$$|M| = \frac{1}{2}\Big(|V(G)| - \mathrm{odd}(G - T) + |T| \Big) .$$

   (b) Explain how you know M is a maximum matching.

   (c) Determine the avoidable and unavoidable vertices in G.

4. Prove that the Perfect Matching Criterion implies the Maximum Matching Formula. *(Hint: Let $k = \max\{\mathrm{odd}(G-S) - |S| \ : \ S \subseteq V(G)\}$. Let G' be the graph obtained from G by adding k new vertices, all joined to every vertex of G. Use the Perfect Matching Criterion to show G' has a perfect matching M. Deduce that G has a matching of size $|M| - k$. Show this is at least $\frac{1}{2}\{\min\{|V(G)| - \mathrm{odd}(G - S) + |S| : S \subseteq V(G)\}$.)*



5. Let G be a bipartite graph with bipartition (X, Y). For a subset S of X, $N(S)$ denotes all the vertices in Y adjacent to at least one vertex in S — the "neighbours" of S. Use the Maximum Matching Formula to prove that the size of a maximum matching in G is equal to

$$|X| - \max\{|S| - |N(S)| \ : \ S \subseteq X\}\,.$$

(**Remark:** I am not interested in other proofs of this fact. It is proved in Math239 by quite different methods. The point is to show that the theorem for general graphs implies the theorem for bipartite graphs.)

(Hint: Let T be a subset of $V(G)$ so that

$$\nu(G) = \frac{1}{2}\Big(|V(G)| - \mathrm{odd}(G - T) + |T|\Big)\,.$$

Find the required set $S \subseteq X$ from among the odd components of $G-T$.)