CO342 ASSIGNMENT #2

The *Maximum Matching Formula* is:

$$\nu(G) = \frac{1}{2}\min\left\{|V(G)| - \text{odd}(G-S) + |S| \ : \ S \subseteq V(G)\right\}.$$

The *Perfect Matching Criterion* is: G has a perfect matching if and only if, for every $S \subseteq V(G)$,

$$\text{odd}(G-S) \leq |S|.$$

1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be sets of (finite) sets with the property that, for every $A \in \mathfrak{A}$ and every $B \in \mathfrak{B}$, $|A| \leq |B|$. Suppose that there is a set $A^* \in \mathfrak{A}$ and a set $B_* \in \mathfrak{B}$ so that $|A^*| = |B_*|$. Prove that

$$|A^*| = \max\{|A| : A \in \mathfrak{A}\} \text{ and } |B_*| = \min\{|B| : B \in \mathfrak{B}\}.$$

SOLUTION. *Let $A \in \mathfrak{A}$. The assumption implies that $|A| \leq |B_*|$. Since $|A^*| = |B_*|$, we see that $|A| \leq |A^*|$ and, therefore, $|A^*| = \max\{|A| : A \in \mathfrak{A}\}$.*

Similarly for $B \in \mathfrak{B}$: $|B| \geq |A^*| = |B_*|$, so $|B_*| = \min\{|B| : B \in \mathfrak{B}\}$.

2. Let G be a graph and let $S \subseteq V(G)$. Recall $\text{odd}(G-S)$ is the number of components of $G-S$ that have an odd number of vertices. Let M be an matching in G. Prove that

$$|M| \leq \frac{1}{2}\left(|V(G)| - \text{odd}(G-S) + |S|\right).$$

SOLUTION. *Let K be a component of $G-S$. Let M_K consist of those edges in M having both ends in K. Clearly, $|M_K| \leq \frac{|V(K)|}{2}$. Each vertex in S is incident with at most one edge of M. Let M^S denote the edges of M incident with vertices in S, so $|M^S| \leq |S|$.*

Clearly, every edge in M is either in M^S or in one of the M_K. Therefore, $|M| = |M^S| + \sum_K |M_K|$, where the sum is over all the components K of $G-S$.

Break up the sum $\sum_K |M_K|$ into the two sums

$$\sum_{K,|V(K)| \text{ even}} |M_K| + \sum_{K,|V(K)| \text{ odd}} |M_K|.$$

Thus,

$$\sum_K |M_K| \leq \sum_{K,|V(K)|\ even} \frac{|V(K)|}{2} + \sum_{K,|V(K)|\ odd} \frac{|V(K)|-1}{2}$$
$$= \frac{|V(G-S)| - \text{odd}(G-S)}{2}.$$

It follows that

$$|M| \leq |S| + \frac{|V(G-S)| - \text{odd}(G-S)}{2},$$

which is the same as the inequality in the question.

3. Let G be the graph in the figure and let T consist of the three red vertices.

   (a) Find a matching M in G so that

   $$|M| = \frac{1}{2}\bigg(|V(G)| - \text{odd}(G-T) + |T|\bigg).$$

   SOLUTION. *My matching M^* consisting of the green edges in the figure has 9 edges. In this case, $|V(G)| = 22$, $\text{odd}(G-T) = 7$, and $|T| = 3$. Therefore, $\frac{1}{2}\bigg(|V(G)| - \text{odd}(G-T) + |T|\bigg) = \frac{1}{2}(22 - 7 + 3) = 9$, as required.*

   (b) Explain how you know M is a maximum matching.

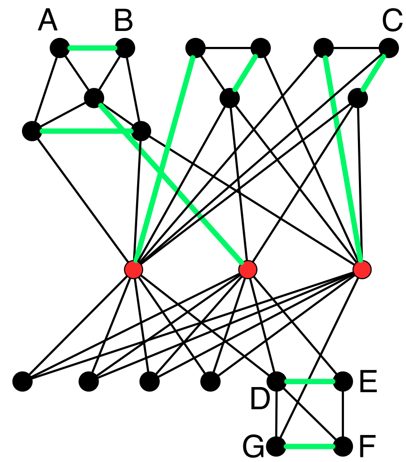
   SOLUTION. *From (2), for any matching M and any $S \subseteq V(G)$,*

   $$|M| \leq \frac{1}{2}\left(|V(G)| - \text{odd}(G-S) + |S|\right).$$

   From (1), M^* is a maximum matching.

   (c) Determine the avoidable and unavoidable vertices in G.

   SOLUTION. *The unavoidable vertices are A, B, C, D, E, F, and G, as well as the three vertices in T. All other vertices are avoidable.*



4. Prove that the Perfect Matching Criterion implies the Maximum Matching Formula. *(Hint: Let* $k = \max\{\mathrm{odd}(G-S) - |S| \ : \ S \subseteq V(G)\}$*. Let* G' *be the graph obtained from* G *by adding* k *new vertices, all joined to every vertex of* G*. Use the Perfect Matching Criterion to show* G' *has a perfect matching* M*. Deduce that* G *has a matching of size* $|M| - k$*. Show this is at least* $\frac{1}{2}\{\min\{|V(G)| - \mathrm{odd}(G-S) + |S| : S \subseteq V(G)\}$*.)*

   SOLUTION. *Let* k *and* G' *be given as in the hint. Let* W *consist of the* k *new vertices. Let* $S \subseteq V(G')$*. We show that* $|S| \geq \mathrm{odd}(G' - S)$*. This holds if* $k = 0$*, in which case the Perfect Matching Theorem implies* G *has a perfect matching, which easily combines with* $k = 0$ *to show that* $\nu(G) = (1/2)\min\{|V(G)| - \mathrm{odd}(G-S) + |S| : S \subseteq V(G)\}$*.*

   We therefore suppose $k > 0$*. We claim* G' *has only one component and an even number of vertices. We first note that, for any set* $S \subseteq V(G)$*,* $|V(G)|$ *and* $\mathrm{odd}(G-S) - |S|$ *are either both even or both odd. To see this,* $|V(G)|$ *is the sum of* $|S|$ *and the numbers of vertices in each component of* $G - S$*. Taken modulo 2, the even components of* $G - S$ *contribute nothing to the sum, while each odd component contributes 1. This is the same as saying that* $|V(G)| \equiv \mathrm{odd}(G-S) + |S|$ *(mod 2). This is equivalent to* $|V(G)| \equiv \mathrm{odd}(G-S) - |S|$ *(mod 2), as required.*

   So we have proved that, for any $S \subseteq V(G)$*,* $\mathrm{odd}(G-S) - |S| \equiv |V(G)|$ *(mod 2). Since* k *is equal to* $\mathrm{odd}(G-T) - |T|$*, for some* $T \subseteq V(G)$*,* $|V(G)| \equiv k$ *(mod 2). That is,* $|V(G')| = |V(G)| + k$ *is* *even. Moreover, since* $k \geq 1$, G' *has more vertices than* G *does. Every vertex of* G' *is joined to any particular vertex of* G' *by a path of length at most 2, so* G' *is connected.*

To show that, for every $S \subseteq V(G')$, $|S| \geq \text{odd}(G' - S)$, *there are three cases: (i)* $W \subseteq S$*; (ii)* $V(G) \subseteq S$*; and (iii) some vertex of* W *is not in* S *and some vertex of* $V(G)$ *is not in* S.

For (i), let $S' = S \setminus W$. *Then* $|S| = |S'| + k$ *and* $S' \subseteq V(G)$. *Also,* $G' - S = G - S'$, *so* $\text{odd}(G' - S) = \text{odd}(G - S')$. *Therefore,* $|S| - \text{odd}(G' - S) = |S'| + k - \text{odd}(G - S')$. *Since* $k = \max\{\text{odd}(G - T) - |T| : T \subseteq V(G)\}$, *we see that* $k \geq \text{odd}(G - S') - |S'|$. *Thus,* $|S| - \text{odd}(G' - S) \geq 0$, *as required.*

In case (ii), $G' - S$ *consists of those vertices in* W *that are not in* S. *It is the definition that* $k = |W|$*; we claim that* $|V(G)| \geq |W|$. *To see this, let* T *be a subset of* $V(G)$ *so that* $k = \text{odd}(G - T) - |T|$. *Then there are at least* k *odd components of* $G - T$, *each of which has a different vertex of* G. *Therefore,* G *has at least* k *vertices.*

Now $\text{odd}(G' - S) = |W \setminus S|$. *Note that* $|S| = |V(G)| + k - |W \setminus S|$. *Thus,*

$$
\begin{aligned}
|S| &= |V(G)| + k - |W \setminus S| \\
&\geq |V(G)| + k - |W| \\
&\geq |V(G)| \\
&\geq |W| \\
&\geq |W \setminus S| \\
&= \text{odd}(G' - S)\,.
\end{aligned}
$$

Finally, in case (iii), let $v \in V(G) \setminus S$ *and* $w \in W \setminus S$. *By the construction of* G', vw *is an edge of* G'. *Therefore,* $G' - S$ *is connected. Thus, if* $|S| \geq 1$, *it is obvious that* $|S| \geq \text{odd}(G' - S)$. *If, on the other hand,* $|S| = 0$, *then* $G' - S = G'$*; since* G' *is connected and has an even number of vertices,* $\text{odd}(G') = 0$ *and so, when* $|S| = 0$, $|S| \geq \text{odd}(G' - S)$ *also holds.*

The Perfect Matching Criterion shows that G' *has a perfect matching* M'. *If we delete the vertices of* W *from* G' *and their incident edges from* M', *we get a matching* M *in* G *of size precisely* $|M'| - k$, *which* *is* $(1/2)|V(G')| - k$*, or* $(|V(G)| - k)/2$*. Thus,*

$$\begin{aligned} |M| &= \frac{1}{2}\left(|V(G)| - k\right) \\ &= \frac{1}{2}\left(|V(G)| - \max\{\mathrm{odd}(G-S) - |S| : S \subseteq V(G)\}\right) \\ &= \frac{1}{2}\min\left\{|V(G)| - \mathrm{odd}(G-S) + |S| : S \subseteq V(G)\right\}, \end{aligned}$$

as required.

5. Let G be a bipartite graph with bipartition (X, Y). For a subset S of X, $N(S)$ denotes all the vertices in Y adjacent to at least one vertex in S — the "neighbours" of S. Use the Maximum Matching Formula to prove that the size of a maximum matching in G is equal to

$$|X| - \max\{|S| - |N(S)| \ : \ S \subseteq X\}.$$

(**Remark:** I am not interested in other proofs of this fact. It is proved in Math239 by quite different methods. The point is to show that the theorem for general graphs implies the theorem for bipartite graphs.)

SOLUTION. *Let* T *be as in the hint. Then* $G - T$ *has* $\mathrm{odd}(G-T)$ *odd components, plus some even components. Let* M *be a maximum matching in* G*. Then* M *provides a perfect matching for every even component of* $G - T$ *and, within each odd component,* M *saturates all but one of the vertices in the odd component with edges contained in the odd component. Every vertex in* T *is saturated by an edge of* M *whose other end is in some odd component of* $G - T$*.*

Let $\mathcal{X}$ *be the set of the odd components of* $G-T$ *having more vertices in* X *than in* Y *and let* S *be the set of vertices in* X *in those components. Notice that the neighbours of* S *are the vertices in* Y *that are in either* T *or an odd component in* $\mathcal{X}$*.*

It v *is a vertex in* $Y \cap T$*, then* v *is incident with an edge* vw *of* M *and* w *is in and odd component of* $G - T$*. Because* G *is bipartite,* $w \in X$ *and, therefore,* $w \in S$*. In particular, every vertex of* $Y \cap T$ *is in* $N(S)$*.*

Combining all the above observations, M *saturates every vertex in* X *except exactly* $|\mathcal{X}| - |X \cap T|$ *vertices. We rewrite this last expression in terms of* $|S|$ *and* $|N(S)|$*.*

Every component C in $\mathcal{X}$ has $(|V(C)|+1)/2$ vertices in X and $(|V(C)|-1)/2$ vertices in Y. That is, $|S|$ is the sum the numbers $(|V(C)|+1)/2$, over all the components C in $\mathcal{X}$. On the other hand, $|N(S)|$ is the sum of the numbers $(|V(C)|-1)/2$ over all those same components, plus $|Y \cap T|$. The difference $|S| - |N(S)|$ is, therefore, the sum of

$$\frac{|V(C)|+1}{2} - \frac{|V(C)|-1}{2}$$

over all the components in $\mathcal{X}$, minus $|Y \cap T|$. Since the displayed numbers are all 1's, $|S| - |N(S)| = |\mathcal{X}| - |Y \cap T|$.

Since every matching must miss at least $|S| - |N(S)|$ vertices in X, M is a maximum matching and $|S| - |N(S)|$ maximizes the estimate of the number of missed vertices. That is, $|M| = |X| - |S| + |N(S)| = |X| - \max\{|T| - |N(T)| : T \subseteq X\}$, as required.