CO342 ASSIGNMENT #3
DUE: IN CLASS, WEDNESDAY 25 MAY 2011

1. Let G be a graph and let H and K be connected subgraphs of G. Suppose there is a vertex of G in both H and K. Prove that $H \cup K$ is connected.

2. Let G be a graph and let $\mathcal{P}$ be some set of subgraphs of G. An element H of $\mathcal{P}$ is *$\mathcal{P}$-maximal* if there is no graph K in $\mathcal{P}$ so that H is a proper subgraph of K (that is, $H \subseteq K$ and $H \neq K$).

   Prove that if $H \in \mathcal{P}$, then there is a $\mathcal{P}$-maximal element K of $\mathcal{P}$ so that $H \subseteq K$.

3. Let G be a connected graph. A *cut-vertex in G* is a vertex v of G so that $G - v$ is not connected. Let $\mathcal{Q}$ denote the set of subgraphs H of G so that there is no cut-vertex in H.

   (a) Let uv be any edge of G. Let H_{uv} denote the subgraph of G consisting of just u, v, and uv. Show that there is no cut-vertex in H_{uv}.

   (b) Using Question 2 or otherwise, prove that there is a $\mathcal{Q}$-maximal subgraph of G containing uv.

   (c) Let H and K be subgraphs of G that are in $\mathcal{Q}$. Show that if H and K have at least two vertices in common, then the union of H and K is also in $\mathcal{Q}$.

   (d) Show that if H and K are distinct $\mathcal{Q}$-maximal subgraphs of G, then H and K have at most one vertex in common.

4. The *blocks* of a graph G are the $\mathcal{Q}$-maximal subgraphs of G. Prove that:

   (a) every edge is in a unique block of G

   (b) if uv is an edge of G, then the subgraph of G consisting of just u, v, and uv is a block of G if and only if uv is a bridge of G.

5. Let G be a graph and let T be a spanning tree of G.

   (a) Determine which vertices of T are cut-vertices of T and which are not.

(b) Prove that every cut-vertex of G is a cut-vertex of T.

(c) Deduce that G has at least two vertices that are not cut-vertices of G.