CO342 ASSIGNMENT #3 SOLUTIONS

1. Let G be a graph and let H and K be connected subgraphs of G. Suppose there is a vertex of G in both H and K. Prove that $H \cup K$ is connected.

   SOLUTION. *Let u be a vertex of G that is in both H and K. For any vertices v and w of $H \cup K$, there is a uv-path P_v in $H \cup K$ (actually either in H or K, depending on where v is) and (similarly) a uw-path P_w in $H \cup K$. Therefore there is a vw-walk in $H \cup K$, and so there is a vw-path in $H \cup K$.*

2. Let G be a graph and let $\mathcal{P}$ be some set of subgraphs of G. An element H of $\mathcal{P}$ is $\mathcal{P}$-*maximal* if there is no graph K in $\mathcal{P}$ so that H is a proper subgraph of K (that is, $H \subseteq K$ and $H \neq K$).

   Prove that if $H \in \mathcal{P}$, then there is a $\mathcal{P}$-maximal element K of $\mathcal{P}$ so that $H \subseteq K$.

   SOLUTION. I have intentionally made this quite formal. *Let $H_0 = H$. For a given non-negative integer i, we suppose inductively that we have a sequence $H_0, H_1, \dots, H_i$ of graphs in $\mathcal{P}$ so that, for each $j \in \{1, 2, \dots, i\}$, H_{j-1} is a proper subgraph of H_j. (With our definition of H_0, we have this for $i = 0$.)*

   If H_i is $\mathcal{P}$-maximal, then we are done: $H \subseteq H_i$. If H_i is not $\mathcal{P}$-maximal, then it is a proper subgraph of some $H_{i+1} \in \mathcal{P}$ and the sequence grows longer.

   The crucial observation — trivially proved by induction on i — is that, for every $i \geq 0$,

$$|V(H)| + |E(H)| + i \leq |V(H_i)| + |E(H_i)| \leq |V(G)| + |E(G)| \,.$$

   Therefore, i can be at most

$$(|V(G)| + |E(G)|) - (|V(H)| + |E(H)|) \ ,$$

   so, for some $i \leq (|V(G)| + |E(G)|) - (|V(H)| + |E(H)|)$, H_i is $\mathcal{P}$-maximal, as required.

3. Let G be a connected graph. A *cut-vertex in G* is a vertex v of G so that $G - v$ is not connected. Let $\mathcal{Q}$ denote the set of **(inserted in the solutions, but not in the original question** CONNECTED subgraphs H of G so that there is no cut-vertex in H.

   (a) Let uv be any edge of G. Let H_{uv} denote the subgraph of G consisting of just u, v, and uv. Show that there is no cut-vertex in H_{uv}.

   SOLUTION. *Deleting a vertex of H_{uv} leaves a graph with one vertex, which is necessarily connected. Therefore, H_{uv} has no cut-vertex.*

   (b) Using Question 2 or otherwise, prove that there is a $\mathcal{Q}$-maximal subgraph of G containing uv.

   SOLUTION. *Since the preceding part shows H_{uv} is in $\mathcal{Q}$, Question 2 implies there is a $\mathcal{Q}$-maximal element containing H_{uv}.*

   (c) Let H and K be subgraphs of G that are in $\mathcal{Q}$. Show that if H and K have at least two vertices in common, then the union of H and K is also in $\mathcal{Q}$.

   SOLUTION. *Let u and u' be two vertices common to H and K. Let v be any vertex of $H \cup K$.*
   By Question 1, $H \cup K$ is connected. Since H and K are both in $\mathcal{Q}$, both $H - v$ and $K - v$ are connnected. (If, for example, v is not in H, then $H - v$ is just H.) At least one of u and u' is not equal to v, so Question 1 implies $(H - v) \cup (K - v)$ is connected, showing v is not a cut-vertex.

   (d) Show that if H and K are distinct $\mathcal{Q}$-maximal subgraphs of G, then H and K have at most one vertex in common.

   SOLUTION. *If H and K had two vertices in common, then the preceding part shows that $H \cup K \in \mathcal{Q}$. Since H and K are distinct connected subgraphs of G, there is an edge e of G that is in one (say H) and not the other (this would be K). Evidently $K \subseteq H \cup K$, and e is in $H \cup K$ but not in K, so K is a proper subgraph of $H \cup K$. But this contradicts the $\mathcal{Q}$-maximality of K.*

4. The *blocks* of a **CONNECTED (inserted)** graph G are the $\mathcal{Q}$-maximal subgraphs of G. Prove that:

(a) every edge is in a unique block of G,

SOLUTION. *Let uv be any edge of G. Part (a) of Question 3 shows uv is in at least one block. If uv is in two blocks H and K of G, then H and K both contain u and v, contradicting (d) of Question 3. Therefore, uv is in exactly one block of G.*

(b) if uv is an edge of G, then the subgraph of G consisting of just u, v, and uv is a block of G if and only if uv is a bridge of G.

SOLUTION. *Let H_{uv} be the subgraph of G consisting of just u, v, and uv.*

Suppose first that uv is a bridge of G. Then u and v are in different components of $G - uv$. If G has no other vertex, then G is just u, v, and uv; in this case uv is $\mathcal{Q}$-maximal and so is a block of G. Otherwise, let K be a connected subgraph of G containing uv and having a vertex w other than u and v.

We may assume w is in the component K_u of $K - uv$ containing u. This implies that v and w are in different components of $K - u$, showing u is a cut-vertex of K. Therefore, uv is not in any larger connected subgraph that has no cut-vertex. That is, H_{uv} is $\mathcal{Q}$-maximal, so H_{uv} is a block of G.

Conversely, suppose uv is not a bridge of G. Then $G - uv$ is connected, so there is a uv-path P in $G - uv$. Now $P + uv$ is a cycle in G containing uv. Since $P + uv$ has no cut-vertex, H_{uv} is not $\mathcal{Q}$-maximal; that is, H_{uv} is not a block.

5. Let G be a graph and let T be a spanning tree of G.

(a) Determine which vertices of T are cut-vertices of T and which are not.

SOLUTION. *If v is a leaf of T, then v is a cut-vertex; every other vertex of T is a cut-vertex.*

If v is not a leaf, then $T - v$ has deg(v) components, where deg(v) is the number of neighbours of v in T. Therefore, v is not a cut-vertex if and only if v has degree 1 in T; that is, if and only if v is a leaf of T.

(b) Prove that every cut-vertex of G is a cut-vertex of T.

SOLUTION. *Let v be a cut-vertex of G and let u and w be any two vertices of G other than u that are in distinct components of $G - v$. There is no uw-path in $G - v$. Any uw-path in $T - v$ is a uw-path in G, so there is also no uw-path in $T - v$, so $T - v$ is not connected.*

(c) Deduce that G has at least two vertices that are not cut-vertices of G.

SOLUTION. *As long as G has at least two vertices this is true. Any spanning tree T of G has at least two leaves. By Part (a), these leaves are not cut-vertices of T. By Part (b), they are not cut-vertices of G.*