CO342 ASSIGNMENT #4
DUE: IN CLASS WEDNESDAY 1 JUNE 2011

1. Let $\approx$ be the relation on the set $E(G)$ of edges of G defined by: if e and f are in $E(G)$, then $e \approx f$ means either $e = f$ or there is a cycle of G containing both e and f.

   Prove that $\approx$ is a transitive relation. (What you need to prove is: if e, f, and g are edges so that $e \approx f$ and $f \approx g$, then $e \approx g$.)

   SOLUTION. If $e = f$, or $f = g$, or $e = g$, then $e \approx g$ is trivial. Thus, we may assume that e, f, and g are all distinct. Because $e \approx f$, there is a cycle $C_{e,f}$ containing both e and f, and likewise there is a cycle $C_{f,g}$ containing both f and g. We may assume e is not in $C_{f,g}$ and g is not in $C_{e,f}$, as otherwise one of these cycles contains both e and g, as desired.

   In $C_{f,g} - \{f, g\}$, there are two paths; let P be the one from one end a of g to one end c of f, and let Q be the other that joins the end b of g to the end d of f. Since c is in f, c is in $C_{e,f}$, so as we traverse P from a towards c, there is a first vertex c' of P that is in $C_{e,f}$; let P' be the subpath of P from a to c'. Likewise, there is a subpath Q' of Q from b to the first vertex d' of Q that is in $C_{e,f}$.

   The vertices c' and d' are both in $C_{e,f}$ and, furthermore, they are distinct (because c and d are distinct). Let R be the one of the two $c'd'$-paths in $C_{e,f}$ that contains e. Then $(P' \cup Q' \cup R) + g$ is a cycle containing both e and g.

2. Recall that the *blocks* of a connected graph G were defined on Assignment 3 to be the maximal elements of the **CONNECTED** subgraphs of G that do not have a cut-vertex. Prove that if e and f are edges in different blocks of G, then there is no cycle in G containing both e and f.

   SOLUTION. If C is a cycle in G, then C contains no vertex that is a cut-vertex of C. Therefore, C is connected and has no cut-vertex. Thus, there is a maximal subgraph B of G that is connected, has no cut-vertex, and $C \subseteq B$. By definition, B is a block of G.

   We have shown that every cycle is contained in a block of G; therefore, edges in different blocks of G cannot be in the same cycle of G.

3. **Bonus Question; not required** Prove that an equivalence class of $\approx$ is precisely the edge set of a block.

   SOLUTION. It turns out that this is in the notes. Here goes anyway. Question 1 implies that two edges in the same equivalence class are together in some cycle. Question 2 implies this cycle, and therefore the two edges, are contained in the same block.

   Thus, it remains to show that two edges in a single block are equivalent, that is, they are in a cycle together. A connected graph having at least two edges and having no cut-vertex is necessarily 2-connected. In class we proved that any two vertices in a 2-connected graph are together in a cycle. Let e and f be any two edges of G. Then Question 4 implies, if we subdivide both e and f to get new vertices u_e and u_f, then the result is still 2-connected. So there is a cycle in this graph through both u_e and u_f. This corresponds to a cycle in the original graph through both e and f. Therefore, $e \approx f$, as required.

4. Let G be a 2-connected graph and let uv be any edge of G. Let G' be the graph obtained from G by deleting the edge uv and adding a new vertex w that is adjacent just to u and v. (This is called *subdividing the edge uv*.) Prove that G' is 2-connected.

   SOLUTION. Since G is 2-connected, $|V(G)| \geq 3$. Clearly, $|V(G')| = |V(G)| + 1$, so $|V(G')| \geq 4 \geq 3$.

   We must also show that, for any vertex x of G', $G' - x$ is connected. If x is none of u, v, and w, then $G' - x$ is the same as $G - x$, with u and v subdivided. Since G is 2-connected, $G - x$ is connected. If a and b are joined by a path P in $G - x$, then either e is not in P, in which case a and b are joined by P in $G' - x$, or e is in P and we can replace it with (u, w, v) to get a and b joined by a path in $G' - x$. Moreover, since w is adjacent to u in $G' - x$, we see that any two vertices of $G' - x$ are joined to u by paths and, therefore, $G' - x$ is connected.

   If $x = u$ (the case $x = v$ is the same), then $G - u$ is connected, and every vertex of $G - u$ is joined to v by a path in $G - u$ that does not use e; these paths are also in $G' - u$. Also w is joined to v by a path in $G' - u$.

   Finally, suppose $x = w$. Then $G' - x = G - e$. Since every vertex of $G - u$ is joined to v by a path in $G - v$ and every vertex of $G - v$ is joined by a path in $G - v$ to u, and there is a third vertex of G, there is also a path in $G - e$ from u to v. That is, $G' - w$ is connected, as required.

For the last two questions, let k be a positive integer and let G be a graph. Recall that G is *k-connected* if:

(a) $|V(G)| \geq k + 1$; and

(b) for each subset W of $V(G)$ with $|W| = k - 1$, $G - W$ is connected.

5. Let k be a positive integer and let G be a k-connected graph. Let $v_1, v_2, \ldots, v_k$ be distinct vertices of G. Create a new graph H from G by adding a new vertex w that is adjacent to precisely $v_1, v_2, \ldots, v_k$. Prove that H is k-connected.

   SOLUTION. Since H has one more vertex than G and G has at least $k+1$ vertices, H has at least $k+2$ (and therefore at least $k+1$) vertices. So it remains to show that, if W is any set of $k - 1$ vertices in H, then $H - W$ is connected.

   If the new vertex w is in W, then $H - W = G - (W \setminus \{w\})$. We proved in class that a k-connected graph is also $(k - 1)$-connected, so $G - (W \setminus \{w\})$ is connected ($|W \setminus \{w\}| = k - 2$). Therefore, $H - W$ is connected in this case.

   If $w \notin W$, then $G - W$ is connected. Since $|W| = k - 1 < k$, at least one of $v_1, \ldots, v_k$ is not in W, and w is joined to this v_i in $H - W$, showing that $H - W$ is the union of the two connected graphs $(\{w, v_i\}, \{wv_i\})$ and $G - W$. Since these two graphs have v_i in common, $H - W$ is connected.

6. (a) For each integer $n \geq 3$, give an example of a 2-connected graph G_n so that every cycle in G_n contains all the vertices of G_n.

   SOLUTION. The cycle of length n is such an example.

   (b) Prove that if G is a 3-connected graph, then there is a cycle in G that does not contain all the vertices of G.

   SOLUTION. Since G is 3-connected, G has a cycle C. If C does not contain all the vertices of G, then we are done. So suppose C has all the vertices of G. Since G is 3-connected, $|V(G)| \geq 4$. Each vertex in G has degree at least 3 in G, so there is an edge uv

of G that is not in C. But then at least one of the uv-paths in C does not contain all the vertices of G; let P be one such uv-path in C. Then $P + uv$ is a cycle in G that does not contain all the vertices of G.