CO342 ASSIGNMENT #5
DUE: IN CLASS WEDNESDAY 8 JUNE 2011

Midterm is Wed 15 June in class.

1. Let G be a k-connected graph and let $u, v_1, v_2, \ldots, v_k$ be $k+1$ distinct vertices in G. Show that there are k paths $P_1, P_2, \ldots, P_k$ in G so that: (i) for $i = 1, 2, \ldots, k$, P_i joins u and v_i; and (ii) for $i \neq j$, P_i and P_j have only u in common. (Hint: Menger's Theorem that we have just proved and Asst. 4 #5.)

   SOLUTION. *Add a new vertex w adjacent to all of $v_1, v_2, \ldots, v_k$ to get a new graph H. By Asst. 4, #5, H is k-connected. Since v and w are not adjacent in H, Menger's Theorem implies there is a set $\mathcal{P}$ of pairwise internally-disjoint uw-paths in H, with $|\mathcal{P}| = \kappa_H(u, w)$.*

   Consider any set W of vertices for which u and w are in different components of $H - W$. Then $H - W$ is not connected and so, since H is k-connected, $|W| \geq k$. Therefore, $\kappa_H(u, w) \geq k$.

   On the other hand, u and w are in different components of $H - \{v_1, v_2, \ldots, v_k\}$, so $\kappa_H(u, w) \leq k$ and, therefore, $\kappa_H(u, w) = k$. Consequently, $|\mathcal{P}| = k$.

   Moreover, each path in $\mathcal{P}$ contains a v_i. No two paths can contain the same v_i (they are internally-disjoint) and, since there are k paths and k v_i's, each v_i is in exactly one path P_i in $\mathcal{P}$. Letting P_i' be the uv_i-subpath of P_i, we see that $P_1', P_2', \ldots, P_k'$ are the desired paths.

2. Let G be a k-connected graph and let A and B be any two sets of vertices, both having size k. Prove that there are k pairwise **totally** disjoint paths $P_1, P_2, \ldots, P_k$ so that each P_i joins a vertex of A to a vertex of B. (Two paths are totally disjoint if they have no vertices in common.)

   SOLUTION. *Create a new graph H by adding new vertices u and v to G so that u is adjacent precisely the vertices in A and v is adjacent precisely to the vertices in B. By two applications of Asst. 4 #5, H is k-connected. As in the preceding question, there is a set $\mathcal{P}$ of precisely k pairwise internally-disjoint uv-paths in H. Each starts at u, going to a vertex in A, then on to a vertex in B, and thence to v. Take the part from the vertex in A to the vertex in B; the set of these AB-paths is the desired set.*

3. Let G be a graph and k a positive integer. Prove the following.

   (a) Suppose G is $(k+1)$-connected. Prove that, for every vertex v of G, $G - v$ is k-connected.

   SOLUTION. *Note that $|V(G)| \geq k+2$, by definition of $(k+1)$-connected. Therefore $|V(G-v)| = |V(G)| - 1 \geq k+1$, which is one of the two requirements for the k-connection of $G-v$.*

   For the other requirement, let W be any set of vertices in $G-v$ of size $k-1$. Set $W' = W \cup \{v\}$. Since G is $(k+1)$-connected and $|W'| = |W|+1 = k$, $G-W'$ is connected. Therefore, $(G-v)-W = G-W'$ is connected, showing $G-v$ is k-connected.

   (b) Suppose that, for every vertex v of G, $G-v$ is k-connected. Prove that G is $(k+1)$-connected.

   SOLUTION. *Let v be any vertex of G. Then $|V(G)| = |V(G-v)| + 1$. Since $G-v$ is k-connected, $|V(G-v)| \geq k+1$, so $|V(G)| \geq k+2$, one of the requirements for showing G is $(k+1)$-connected.*

   For the other requirement, let W be any subset of $V(G)$ with size k. Let $v \in W$. The graph $G-v$ is k-connected and $W' = W \setminus \{v\}$ is a set of $k-1$ vertices in $G-v$. Thus, $(G-v)-W'$ is connected. But $G-W = (G-v)-W'$, so $G-W$ is connected; this is the other requirement for showing G is $(k+1)$-connected.

4. For each integer $k \geq 2$, find an example of a k-connected graph G_k that has some $k+1$ distinct vertices $v_0, v_1, \ldots, v_k$ that are not all together on a cycle in G_k.

   SOLUTION. *Let G_k be the complete bipartite graph $K_{k,k+1}$. Let $v_0, v_1, \ldots, v_k$ be the vertices in the $k+1$ part of the bipartition.*

   To see that G_k is k-connected, observe that $|V(G_k)| = 2k+1 \geq k+1$. If W is any set of $k-1$ vertices in G_k, then $G-W$ has at least one vertex from each part of the bipartition and, therefore, is connected.

   Because G_k is bipartite, any cycle in G_k has the same number of vertices from each part of the bipartition. Since one part has only k vertices, no cycle can contain all the vertices from the other part; that is, no cycle can contain all of $v_0, v_1, \ldots, v_k$.

5. Let $k \geq 2$ be an integer, let G be a k-connected graph, and let $v_1, v_2, \ldots, v_k$ be distinct vertices of G. Prove that G has a cycle containing all of $v_1, v_2, \ldots, v_k$. (Hint: use induction on k and Asst. 4 #5. For the base case $k = 2$, you may assume the result from class, which proves this for $k = 2$, so I only care about the inductive step. Be careful: it is common to overlook a case in the induction step.)

   SOLUTION. *We are proceeding by induction on k and we are allowed to assume the case $k = 2$. For the inductive step, assume $k \geq 3$ and that the result holds for $k - 1$. Since G is k-connected, we proved in class that G is also $(k-1)$-connected. Applying the inductive result, there is a cycle C in G containing the $k-1$ vertices $v_2, v_3, \ldots, v_k$.*

   If v_1 is also a vertex of C, then we are obviously done: C is a cycle in G containing all of $v_1, v_2, \ldots, v_k$. Therefore, we may assume v_1 is not a vertex of C.

   If there are k vertices, say $v_2, v_3, \ldots, v_k, u$, in C, then using Question 1 we can find k paths $P_1, P_2, P_3, \ldots, P_k$, each joining v_1 to one of $v_2, v_3, \ldots, v_k, u$ and pairwise having only u in common. For each i, let P'_i be the subpath of P_i starting at v_1 and ending at the first vertex of P_i that is in C. Let u'_i denote the vertex of P'_i in C.

   The $k-1$ vertices $v_2, v_3, \ldots, v_k$ occur in some cyclic order $v_{i_2}, v_{i_3}, \ldots, v_{i_{k-1}}$ in C. This means that, for $j = 2, 3, \ldots, k-1$, there is a $v_{i_j}v_{i_{j+1}}$-path C_j in C that does not contain any other one of $v_2, v_3, \ldots, v_k$. Also, there is a $v_{i_k}v_{i_2}$-path C_k that does not contain any other one of $v_2, v_3, \ldots, v_k$. We note that C is the union $C_2 \cup C_3 \cup \cdots \cup C_k$.

   Now the k vertices $u'_1, u'_2, \ldots, u'_k$ are all in C, so some two of them, say u'_r and u'_s, must be in the same C_j. We can now reroute C_j, from v_{i_j} through C_j to the nearer of u'_r and u'_s, then through $P'_r \cup P'_s$ through v_1 to the other of u'_r and u'_s, and finally along C_j to $v_{i_{j+1}}$. Together with the rest of C, we have a cycle through all of $v_1, v_2, \ldots, v_k$.

   If C does not have k vertices, then C has precisely $v_2, v_3, \ldots, v_k$ as its vertices. Now we apply Question 1, but thinking of G as a $(k-1)$-connected graph, to get disjoint (except for v_1) paths from v_1 to each of $v_2, v_3, \ldots, v_k$.

   Now we take C without the edge between some two v_i's, say v_rv_s, and replace the edge with the paths from v_r and v_s to v_1. This gives a cycle *in* G *containing all of* $v_1, v_2, \ldots, v_k$.