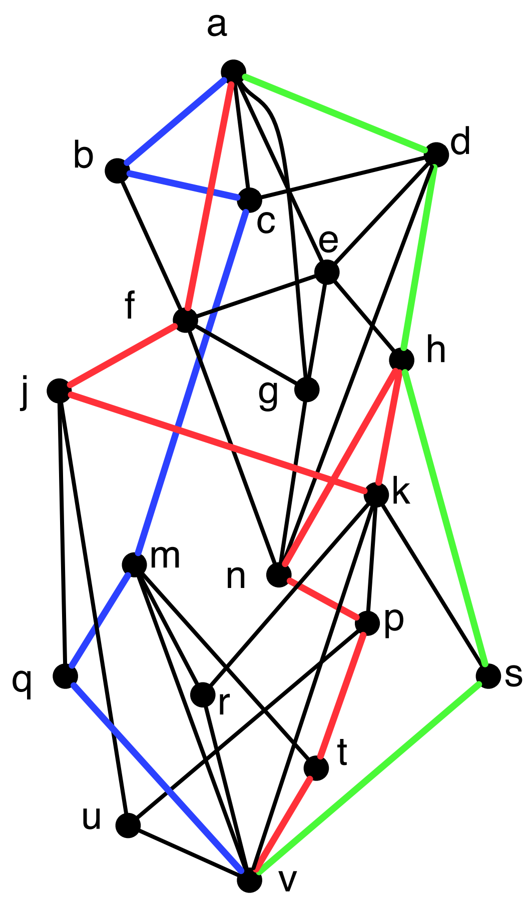
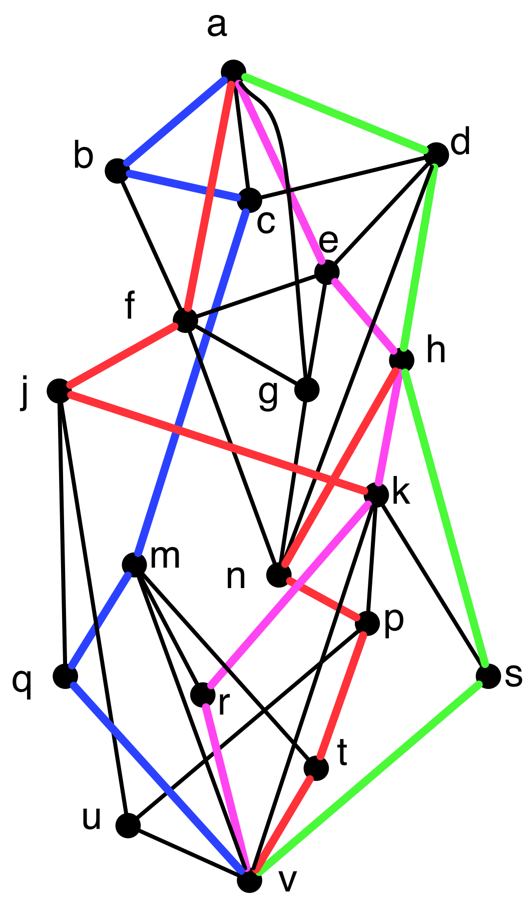
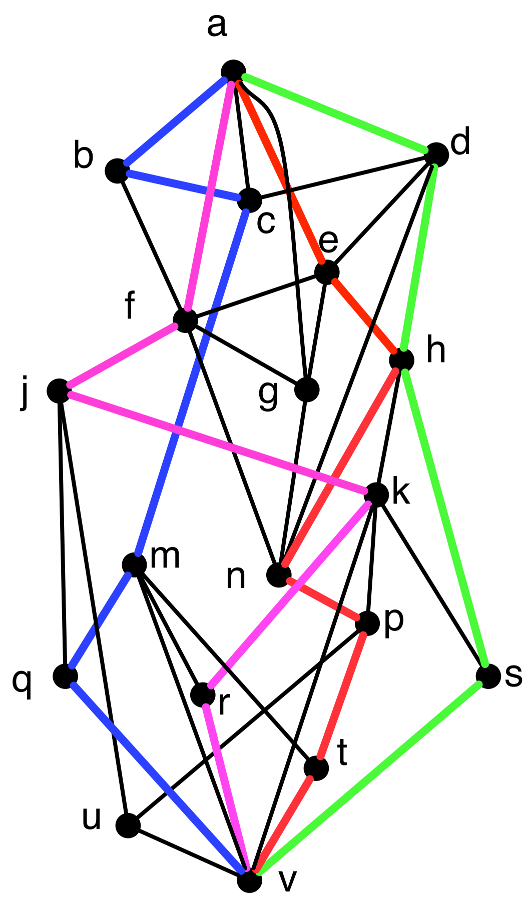
CO342 ASSIGNMENT #6 SOLUTIONS

1. In the figure is an example of a graph G. The set of coloured edges represent three pairwise edge-disjoint av-paths in G. Illustrate the labelling algorithm and use it to produce a largest possible set of pairwise edge-disjoint av-paths in G.

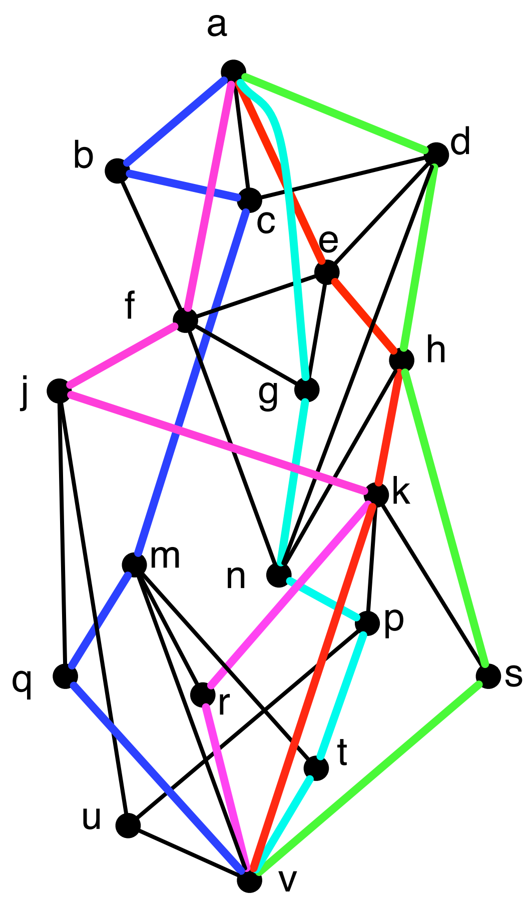




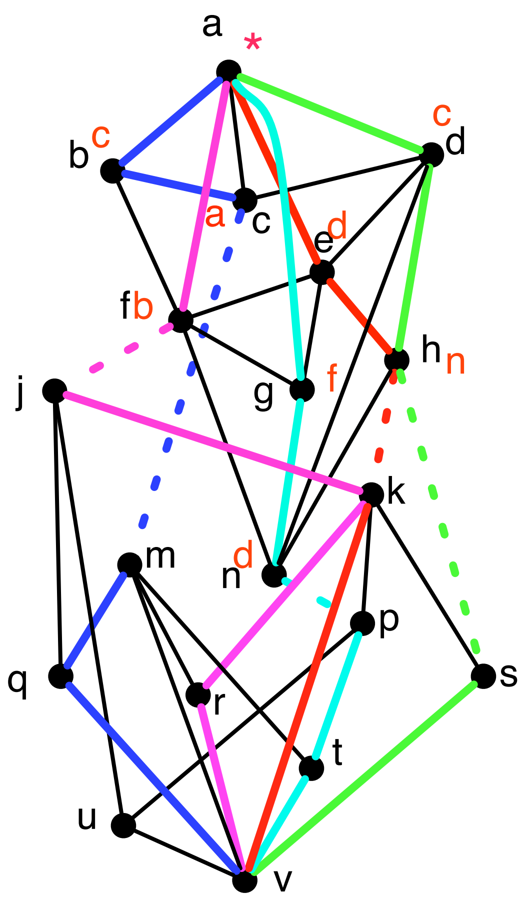
An augmenting path for the original collection is in purple.



The resulting larger collection of edge-disjoint paths.



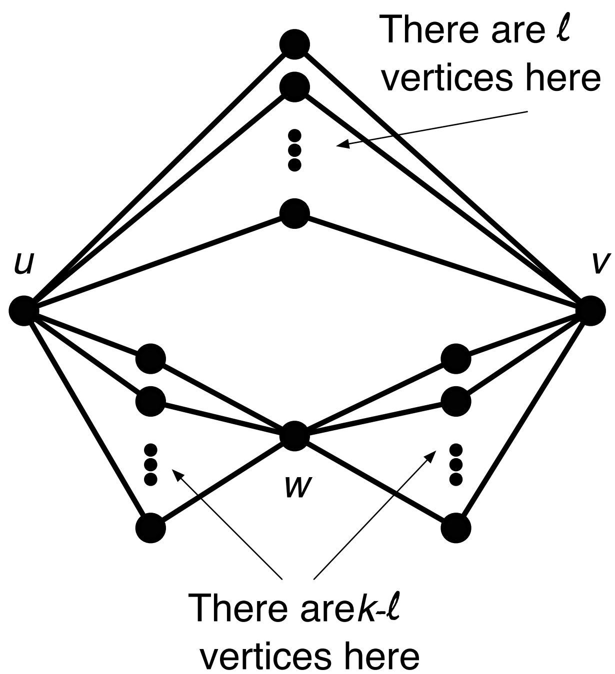
An augmenting path for the new collection is u, g, n, h, k, p, t, v (using the red edge nh) yielding the still larger collection.



The labelled vertices at the end of the labelling algorithm yields the 5-cut shown in dashed edges, one of each of the five non-black colours.

2. For each pair of positive integers k and ℓ, with $k > \ell$, give an example of a graph G having a pair of vertices u and v so that:

   - a largest set of pairwise edge-disjoint uv-paths in G has size k; and
   - there is a vertex w of G, different from u and v, so that the largest set of pairwise edge-disjoint uv-paths in $G - w$ has size ℓ.



3. Let u, v, and w be distinct vertices in a graph G. Suppose there are k pairwise edge-disjoint uv-paths in G and there are k pairwise edge-disjoint vw-paths in G. Prove that there are k pairwise edge-disjoint uw-paths in G.

   SOLUTION. *Let E be a smallest set of edges so that u and w are in different components of $G - E$. Let K_u be the component of $G - E$ containing u and let K_w be the component of $G - E$ containing w.*

   If v is in K_u, then E is a set of edges so that v and w are in different components of $G - E$. Since there are k pairwise edge-disjoint vw-paths *in G, $|E| \geq k$, as required. A very similar argument applies if v is not in K_u.*

4. In this exercise, we shall deduce the edge-disjoint version of Menger's Theorem from the internally disjoint version.

   Internally-disjoint Menger's Theorem: For non-adjacent vertices u and v of G, let $\kappa_G(u, v)$ denote the size of a smallest uv-cut. Then there is a set $\mathcal{P}$ of pairwise internally-disjoint uv-paths in G so that $|\mathcal{P}| = \kappa_G(u, v)$.

   Edge disjoint version of Menger's Theorem: let u and v be vertices in a graph G. Then the maximum size of a set of pairwise edge-disjoint uv-paths in G is equal to the minimum size of a set F of edges of G so that $G - F$ has no uv-path.

   SOLUTION. *Create a new graph G' as follows:*

   - *the vertex set of G' is u, v, and, for each edge e of G, a vertex w_e;*
   - *for each edge e of G incident with u, there is an edge uw_e in G';*
   - *for each edge e of G incident with v, there is an edge vw_e in G';*
   - *if e and e' are edges of G both incident with the vertex x of G, $x \neq u, v$, then w_e and w'_e are adjacent in G'.*

   We apply the internally-disjoint version of Menger's Theorem to G', obtaining a set $\mathcal{P}'$ of internally-disjoint uv-paths in G' and a set W' of vertices in G' so that: (a) u and v are in different components of $G' - W'$; and (b) $|\mathcal{P}'| = |W'|$.

   With one small technical point, we claim that each uv-path in G' produces a uv-path in G. Let $(w_0, w_1, w_2, \ldots, w_k)$ be a uv-path P' in G', so $w_0 = u$ and $w_k = v$. There may be an edge, called a chord, *$w_i w_j$ in G, with $j \geq i+2$, and we could get the shorter path $(w_0, w_1, \ldots, w_i, w_j, w_{j+1}, \ldots, w_k)$. Repeat as long as there is a chord of the current path. Let $P'^* = (v_0, v_1, v_2, \ldots, v_\ell)$ be the resulting path.*

   The path P'^* corresponds to a path in G: start at u, and proceed successively along the edges $v_1, v_2, \ldots, v_\ell$ of G. Since each v_{i-1} is joined by an edge in G' to v_i, this implies that v_{i-1} and v_i have a common end

u_i in G. (The argument is slightly different if $i = 1$, so $v_{i-1} = u$, or $i = \ell$, so $v_\ell = v$, but you should see the point.)

We note that, by choosing P'^*, no other v_j can also be incident in G with u_i, since this implies either $j < i - 1$ and $v_j v_i$ is an edge of G' or $j > i$ and $v_{i-1} v_j$ is an edge of G'. In either case, P'^* has a chord, a contradiction. Therefore, $(u, u_1, u_2, \ldots, u_{k-1}, v)$ is a uv-path P in G.

Thus, each path P' in $\mathcal{P}'$, which we may take to be chordless, produces a path P in G. An edge of P corresponds to an internal vertex of P'. Since the paths in $\mathcal{P}'$ are pairwise internally-disjoint, the paths P are pairwise edge-disjoint.

Also, each vertex of G' in W' corresponds to an edge of G; let E be the set of such edges. If there were a uv-path Q in $G - E$, then, by following the edges of Q, we obtain a uv-path Q' in $G' - W'$. Since there is no such path, we conclude that there is no uv-path in $G - E$. Since $\{P : P' \in \mathcal{P}'\}$ has the same size as $\mathcal{P}'$, and therefore as W', and therefore, as E, we see that this is the edge disjoint version of Menger's Theorem.