CO342 ASSIGNMENT #7
DUE: IN CLASS WEDNESDAY 29 JUNE 2011

1. Let S be a finite set, let X be a subspace of 2^S and let T be any subset of S. Let Y consist of those elements y of X for which $|y \cap T|$ is even.
   - Prove that Y is a subspace of X.
   - Prove that $\dim(Y) \geq \dim(X) - 1$ and that equality holds if and only if there is an element $y' \in X$ so that $|y' \cap T|$ is odd.

2. Let G be a connected graph. The purpose of this exercise is to show that $\dim(\mathcal{Z}(G)) = |E(G)| - |V(G)| + 1$.

   Let $v_1, v_2, \ldots, v_n$ be the vertices of G (so $n = |V(G)|$). We shall construct subspaces $Z_0, Z_1, Z_2, \ldots, Z_{n-1}$ of $2^{E(G)}$ so that:
   - $Z_0 = 2^{E(G)}$;
   - $Z_{n-1} = \mathcal{Z}(G)$; and,
   - for each $i = 1, 2, \ldots, n-1$, $\dim(Z_i) = \dim(Z_{i-1}) - 1$.

   To do this, let E_i be the set of edges incident with v_i and let Z_i denote the subspace of Z_{i-1} consisting of those elements z of Z_{i-1} having $|z \cap E_i|$ even.
   - Use Question 1 to show that, for $i = 1, 2, \ldots, n-1$, $\dim(Z_i) = \dim(Z_{i-1}) - 1$. *(Hint. If $1 \leq i \leq n-1$, then there is a path P in G from v_i to v_n. Show that $E(P) \in \mathcal{Z}_{i-1}$ but $E(P) \notin \mathcal{Z}_i$.)*
   - Show that Z_{n-1} is in fact equal to $\mathcal{Z}(G)$.

3. Let G be a connected graph and let T be a spanning tree of G. For each edge e of G not in T, the subgraph $T + e$ contains a cycle $z_T(e)$.
   - Show that the cycles $z_T(e)$, $e \in E(G) \setminus E(T)$, are linearly independent.
   - How many cycles $z_T(e)$ are there?
   - Use Question 2 to show that the $z_T(e)$, $e \in E(G) \setminus E(T)$, are a basis for $\mathcal{Z}(G)$.

4. Let H be a graph so that every vertex of H has even degree. Prove that either $E(H) = \varnothing$ or there are **pairwise edge-disjoint** cycles $C_1, C_2, \ldots, C_k$ in H so that H is the union $C_1 \cup C_2 \cup \cdots \cup C_k$, plus possibly some isolated vertices. (The number k is not important.) *(Hint: show H has a cycle C_1 and use induction on $H - E(C_1)$.*