CO342 ASSIGNMENT #7

1. Let S be a finite set, let X be a subspace of 2^S and let T be any subset of S. Let Y consist of those elements y of X for which $|y \cap T|$ is even.

   - Prove that Y is a subspace of X.

     SOLUTION. *It suffices to show that $\varnothing \in Y$ and that Y is closed under $\oplus$. For the first, since X is a subspace of 2^S, $\varnothing \in X$. Since $|\varnothing \cap T| = 0$, $\varnothing \in Y$, as required.*
     For the second, we prove the following lemma (which we will use again later).

     Lemma. Let a, b, c be sets. Then

     $$|(a \oplus b) \cap c| \equiv |(a \cap c)| + |(b \cap c)| \pmod 2 .$$

     Proof.

     $$\begin{aligned} x \in (a \oplus b) \cap c &\Leftrightarrow x \in a \oplus b \text{ and } x \in c \\ &\Leftrightarrow x \text{ is in exactly one of } a, b \text{ and } x \in c \\ &\Leftrightarrow x \text{ is in exactly one of } a \cap c \text{ and } b \cap c \\ &\Leftrightarrow x \in (a \cap c) \oplus (b \cap c) . \end{aligned}$$

     It follows that

     $$\begin{aligned} |(a \oplus b) \cap c| &= |(a \cap c) \oplus (b \cap c)| \\ &= |a \cap c| + |b \cap c| - 2|(a \cap c) \cap (b \cap c)| \\ &\equiv |a \cap c| + |b \cap c| \pmod 2 . \qquad \square \end{aligned}$$

     Using the lemma and the fact that both $|y \cap T|$ and $|y' \cap T|$ are even, we conclude that $y \oplus y' \in Y$, as required.

   - Prove that $\dim(Y) \geq \dim(X) - 1$ and that equality holds if and only if there is an element $y' \in X$ so that $|y' \cap T|$ is odd.

     SOLUTION. *If $Y = X$, then $\dim(Y) = \dim(X) \geq \dim(X) - 1$. Moreover, equality does not hold and every $x \in X$ is in Y, so $|x \cap T|$ is even.*

Therefore, we may assume $Y \neq X$. *Since* $Y \subseteq X$, *we conclude that there is an* $x \in X \setminus Y$. *Since* $x \notin Y$, $|x \cap T|$ *is odd. If* $x' \in X \setminus Y$, *then also* $|x' \cap T|$ *is odd. Therefore,* $|(x \oplus x') \cap T|$ *is even (by the lemma above and the fact that both* $|x \cap T|$ *and* $|x' \cap T|$ *are odd). It follows that* $x \oplus x' \in Y$.

In particular, every element of $X \setminus Y$ *is of the form* $y + x$, *with* $y \in Y$. *It follows that* x *extends a basis of* Y *to a basis of* X, *so* $\dim(X) = \dim(Y) + 1$.

2. Let G be a connected graph. The purpose of this exercise is to show that $\dim(\mathcal{Z}(G)) = |E(G)| - |V(G)| + 1$.

   Let $v_1, v_2, \ldots, v_n$ be the vertices of G (so $n = |V(G)|$). We shall construct subspaces $Z_0, Z_1, Z_2, \ldots, Z_{n-1}$ of $2^{E(G)}$ so that:

   - $Z_0 = 2^{E(G)}$;
   - $Z_{n-1} = \mathcal{Z}(G)$; and,
   - for each $i = 1, 2, \ldots, n-1$, $\dim(Z_i) = \dim(Z_{i-1}) - 1$.

   To do this, let E_i be the set of edges incident with v_i and let Z_i denote the subspace of Z_{i-1} consisting of those elements z of Z_{i-1} having $|z \cap E_i|$ even.

   - Use Question 1 to show that, for $i = 1, 2, \ldots, n-1$, $\dim(Z_i) = \dim(Z_{i-1}) - 1$. *(Hint. If* $1 \leq i \leq n-1$, *then there is a path* P *in* G *from* v_i *to* v_n. *Show that* $E(P) \in \mathcal{Z}_{i-1}$ *but* $E(P) \notin \mathcal{Z}_i$.*)*

     SOLUTION. *Question 1 shows that* $\dim(Z_i) \geq \dim(Z_{i-1}) - 1$. *Equality will be established by finding an element of* Z_{i-1} *that is not in* Z_i. *Let* P *be a* $v_i v_n$*-path in* G. *Then exactly one edge of* $E(P)$ *is incident with each of* v_i *and* v_n, *while either 0 or 2 (both even numbers) of the edges of* P *are incident with any other vertex of* G. *Thus* $E(P) \in Z_{i-1}$, *but* $|E(P) \cap E_i|$ *is odd.*

   - Show that Z_{n-1} is in fact equal to $\mathcal{Z}(G)$.

     SOLUTION. *Let* $z \in \mathcal{Z}(G)$. *Then each of the vertices* $v_1, v_2, \ldots, v_{n-1}, v_n$ *is incident with an even number of edges in* z*; in particular this is true for* $v_1, v_2, \ldots, v_{n-1}$, *showing* $z \in Z_{n-1}$. *We conclude that* $\mathcal{Z}(G) \subseteq Z_{n-1}$.

For the reverse containment, let $z \in Z_{n-1}$. Then each of the vertices $v_1, v_2, \ldots, v_{n-1}$ is incident with an even number of edges in z; this leaves the possibility that only v_n is incident with an odd number of edges in z. But this is impossible, since the number of vertices incident with an odd number of edges in z is even. Therefore, every vertex of G is incident with an even number of edges in z, showing $z \in \mathcal{Z}(G)$. Therefore, $Z_{n-1} \subseteq \mathcal{Z}(G)$. Combined with the preceding paragraph, we deduce that $Z_{n-1} = \mathcal{Z}(G)$.

3. Let G be a connected graph and let T be a spanning tree of G. For each edge e of G not in T, the subgraph $T + e$ contains a cycle $z_T(e)$.

   - Show that the cycles $z_T(e)$, $e \in E(G) \setminus E(T)$, are linearly independent.

     SOLUTION. *Let* $\bigoplus_{e \in E(G) \setminus E(T)} \alpha_e z_T(e) = \varnothing$. *If some* $\alpha_e = 1$, *then* $e \in z_T(e)$ *and is not in any other* $z_T(e')$. *Therefore,* e *would be in* $\bigoplus_{e \in E(G) \setminus E(T)} \alpha_e z_T(e)$, *a contradiction. Therefore, every* α_e *is 0, so the* $z_T(e)$ *are linearly independent.*

   - How many cycles $z_T(e)$ are there?

     SOLUTION. *There are* $|E(G)|$ *edges in* G, $|E(T)|$ *of which are in* T. *Therefore, there are* $|E(G)| - |E(T)|$ *cycles* $z_T(e)$. *Since* $|E(T)| = |V(G)| - 1$, *there are* $|E(G)| - |V(G)| + 1$ *cycles* $z_T(e)$.

   - Use Question 2 to show that the $z_T(e)$, $e \in E(G) \setminus E(T)$, are a basis for $\mathcal{Z}(G)$.

     SOLUTION. *By Question 2,* $\dim(\mathcal{Z}(G)) = |E(G)| - |V(G)| + 1$. *Since this is the number of linearly independent* $z_T(e)$, *the* $z_T(e)$ *is a basis for* $\mathcal{Z}(G)$.

4. Let H be a graph so that every vertex of H has even degree. Prove that either $E(H) = \varnothing$ or there are **pairwise edge-disjoint** cycles $C_1, C_2, \ldots, C_k$ in H so that H is the union $C_1 \cup C_2 \cup \cdots \cup C_k$, plus possibly some isolated vertices. (The number k is not important.) *(Hint: show H has a cycle C_1 and use induction on $H - E(C_1)$.*

   SOLUTION. *If H has no edges, then we are done: H consists of isolated vertices.*

For the induction, assume that H has some edges. If H has no cycles, then H is a forest and some component of H is a tree with at least one edge. This component has a vertex that has degree 1 in H; this is impossible as 1 is odd. Therefore, H has a cycle C_1. Now $H - E(C_1)$ is a graph in which every vertex has even degree and $H - E(C_1)$ has fewer edges than H has. By induction, $H - E(C_1)$ is the union of edge-disjoint cycles $C_2, C_3, \ldots, C_k$, plus isolated vertices. Therefore, H is the union of the edge-disjoint cycles $C_1, C_2, \ldots, C_k$, plus isolated vertices.