CO342 ASSIGNMENT #8
DUE: IN CLASS WEDNESDAY 6 JULY 2011

1. Let $n \geq 3$ be an integer. Show that every element of $\mathcal{Z}(K_n)$ is a linear combination of edge-sets of 3-cycles in the complete graph K_n with n vertices.

   SOLUTION. It suffices to show that every cycle in K_n is a linear combination of 3-cycles. We do this on the length n of the cycle C. If $n = 3$, then C is a 3-cycle, then obviously C is a linear combination of 3-cycles. If $n > 3$, then let $C = (x_0, x_1, x_2, \ldots, x_{n-1}, x_0)$. Since $n \geq 4$, x_0x_2 is not an edge of C. Let T be the 3-cycle (x_0, x_1, x_2, x_0). Then $E(C) \oplus E(T)$ is the edge-set of the cycle $(x_0, x_2, x_3, \ldots, x_{n-1}, x_0)$, which has length $n - 1$. By induction, it is a linear combination of 3-cycles, which, together with T, yields a linear combination of 3-cycles that adds up to C.

2. Let G be a graph and let H be a 2-connected subgraph of G. Let P be a path in G with length at least one and such that $P \cap H$ consists of just the ends of P. Prove that $H \cup P$ is 2-connected.

   SOLUTION. Since H is 2-connected, $|V(H)| \geq 3$; therefore, $|V(H \cup P)| \geq |V(H)| \geq$, as required for the first part of showing $H \cup P$ is 2-connected.

   For the main component of 2-connection, let v be any vertex of $H \cup P$. If v is in H, then $H - v$ is connected. If w is any vertex of P that is not in H, then there are two disjoint paths in $H \cup P$ from w to vertices in H, so at least one of these is still present in $(H \cup P) - v$, showing $(H \cup P) - v$ is connected.

   A similar argument holds is v is in P but not in H. This shows that $(H \cup P) - v$ is connected in all cases and, therefore, $H \cup P$ is 2-connected.

3. Let G be a graph and let H be a 2-connected subgraph of G. Let P be a path in G with length at least one and such that $P \cap H$ consists of just the ends of P. Prove that $\dim(\mathcal{Z}(H \cup P)) = 1 + \dim(\mathcal{Z}(H))$.

   SOLUTION. Let Q be any path in H joining the ends of P and let C^* be the cycle $P \cup Q$. Clearly, $E(C^*)$ is in $\mathcal{Z}(H \cup P) \setminus \mathcal{Z}(H)$, so $\dim(\mathcal{Z}(H \cup P)) \geq 1 + \dim(\mathcal{Z}(H))$.

Let C be any cycle in $H \cup P$. Observe that either $P \subseteq C$ or $C \subseteq H$. If $P \subseteq C$, then $E(C) \oplus E(C^*)$ has no edge in P and, therefore, is in $\mathcal{Z}(H)$. Therefore, for every cycle C in $H \cup P$ is either $E(C)$ is in $\mathcal{Z}(H)$ or $E(C) \oplus E(C^*) \in \mathcal{Z}(H)$. Thus, every cycle in $H \cup P$ is a linear combination of C^* and something in $\mathcal{Z}(H)$, showing that everything in the cycle space of $H \cup P$ has this property.

It follows that we need only add C^* to a basis of $\mathcal{Z}(H)$ to get a spanning set for $\mathcal{Z}(H \cup P)$. Thus, $\dim(\mathcal{Z}(H \cup P)) \leq 1 + \dim(\mathcal{Z}(H)$. Combining this with our earlier observation, $\dim(\mathcal{Z}(H \cup P)) = 1 + \dim(\mathcal{Z}(H))$.

4. A *cycle double cover* of a graph G is a sequence $(C_1, C_2, \ldots, C_k)$ of cycles in G so that every edge of G appears in exactly two of the C_i. Suppose $(C_1, C_2, \ldots, C_k)$ is a cycle double cover of a graph G.

   (a) Show that $\bigoplus_{i=1}^{k} E(C_i) = \varnothing$.

   SOLUTION. Each edge of G is in exactly two of $C_1, C_2, \ldots, C_k$ and so is not in $\bigoplus_{i=1}^{k} E(C_i)$. Therefore, $\bigoplus_{i=1}^{k} E(C_i) = \varnothing$.

   (b) Suppose there is a t with $1 < t < k$ so that $\bigoplus_{i=1}^{t} E(C_i) = \varnothing$. Prove:

   - $\bigoplus_{i=t+1}^{k} E(C_i) = \varnothing$; and

     SOLUTION.

     $$\bigoplus_{i=t+1}^{k} E(C_i) = \left(\bigoplus_{i=1}^{k} E(C_i)\right) \oplus \left(\bigoplus_{i=1}^{t} E(C_i)\right) = \varnothing \oplus \varnothing = \varnothing\,.$$

   - if C is a cycle with one edge in C_1 and another edge in C_k, show that $E(C)$ is not in the span of $E(C_1), E(C_2), \ldots, E(C_k)$.

     SOLUTION. Let $A = E(C_1) \cup E(C_2) \cup \cdots \cup E(C_t)$. Since $\bigoplus_{i=1}^{t} E(C_i) = \varnothing$ and $(C_1, C_2, \ldots, C_k)$ is a cycle double cover of G, every edge of A is in precisely two of $C_1, C_2, \ldots, C_t$; every edge of $E(G) \setminus A$ is in precisely two of $C_{t+1}, C_{t+2}, \ldots, C_k$ and $E(C_{t+1}) \cup E(C_{t+2}) \cup \cdots \cup E(C_k) = E(G) \setminus A$.
     If $E(C)$ is in the span of $E(C_1), E(C_2), \ldots, E(C_k)$, then there are scalars α_i so that $E(C) = \bigoplus_{i=1}^{k} \alpha_i E(C_i)$. Let $z_1 = \bigoplus_{i=1}^{t} \alpha_i E(C_i)$ and let $z_2 = \bigoplus_{i=t+1}^{k} \alpha_i E(C_i)$. Evidently, $E(C) = z_1 \oplus z_2$, $z_1 \subseteq A$, and $z_2 \subseteq E(G) \setminus A$. Thus, $E(C)$ is the disjoint union of z_1 and z_2.

Since C has an edge in C_1, $E(C) \cap A \neq \varnothing$; likewise, $E(C) \cap (E(G) \setminus A) \neq \varnothing$. It follows that z_1 is a proper, non-empty, subset of $E(C)$. But every proper subset of $E(C)$ does not contain the edge-set of any cycle, while the non-empty element z_1 of the cycle space of G contains the edge-set of a cycle. This is a contradiction, showing that $E(C)$ cannot be a linear combination of $C_1, C_2, \ldots, C_k$.

5. Let H be a 2-connected subgraph of a graph G and let P be a path in G so that:

   (a) P has both ends in H;

   (b) P is otherwise disjoint from H; and

   (c) $G = H \cup P$.

   Let $(C_1, C_2, \ldots, C_k)$ be a cycle double cover of G so that $\{C_1, C_2, \ldots, C_k\}$ spans the cycle space of G. If P is contained in both C_1 and C_2, but not in any other C_i, prove that $\{C_3, C_4, \ldots, C_k\}$ spans the cycle space of H.

   SOLUTION. We remark that no edge of P is in any of $C_3, C_4, \ldots, C_k$, so these cycles are all in H.

   Let $z \in \mathcal{Z}(H)$. Then $Z \in \mathcal{Z}(G)$, so there are scalars α_i for which $z = \bigoplus_{i=1}^k \alpha_i E(C_i)$. Since no edge of P is in z, $\alpha_1 = \alpha_2$. If these are both 0, then $z = \bigoplus_{i=3}^k \alpha_i E(C_i)$, so z is in the span of $C_3, C_4, \ldots, C_k$ as required.

   On the other hand, if both α_1 and α_2 are 1, then we note that $\bigoplus_{i=1}^k E(C_i) = \varnothing$. Therefore, $E(C_1) \oplus E(C_2) = \bigoplus_{i=3}^k E(C_i)$. Now we have

$$
\begin{aligned}
z &= E(C_1) \oplus E(C_2) \oplus \left(\bigoplus_{i=3}^{k} \alpha_i E(C_i) \right) \\
&= \left(\bigoplus_{i=3}^{k} E(C_i) \right) \oplus \left(\bigoplus_{i=3}^{k} \alpha_i E(C_i) \right) \\
&= \left(\bigoplus_{i=3}^{k} (1 + \alpha_i) E(C_i) \right),
\end{aligned}
$$

   as required.