CO342 ASSIGNMENT #9
DUE: IN CLASS WEDNESDAY 13 JULY 2011

Throughout this assignment, we will be dealing with *peripheral cycles*. Let C be a cycle in a graph G.

Definition. 1. A *chord* of C is an edge e of G that is not in C, but both ends of e are in C.

2. C is a *peripheral cycle* if C has no chords and $G - V(C)$ is connected.

1. Determine (with some justification) which cycles in K_n, $K_{m,n}$, and Q_n are chordless and which are peripheral.

   SOLUTION. *If C is a cycle in K_n of length at least 4, then any vertices of C not adjacent in C are adjacent in K_n and that edge is a chord of C. Thus, the only cycles in K_n that are chordless are the ones of length 3. These are also peripheral (at least for $n \geq 4$), as the complement of any 3-cycle is connected.*

   Similarly, in $K_{m,n}$, the 4-cycles are the chordless and peripheral cycles.

   The question for Q_n is more complicated. For $n = 3$, the answer is precisely the 4-cycles; for $n \geq 5$, the 4-cycles are chordless and peripheral. But for $n = 4$, $(0000, 0001, 0011, 0111, 0110, 0100)$ is a chordless and peripheral cycle and there are many others. On the other hand, $(0000, 0001, 0011, 0111, 0101, 0100, 0000)$ is a 6-cycle with the chord 0001-0101 and so it is not peripheral. If anyone makes serious progress on this, they should get bonus marks.

2. Let G be a graph embedded in the plane and let C be a peripheral cycle of G. Prove that C bounds a face of G.

   SOLUTION. *If C does not bound a face it is because part of G is embedded inside C and part of G is embedded outside C. Suppose there is a vertex v_i inside C and a vertex v_o outside C. Since C is peripheral, $G - V(C)$ is connected, so there is a v_iv_o-path P in G that is totally disjoint from C.*

   In the plane, the path P must cross C, which is impossible in a planar embedding. Therefore, the vertices v_i and v_o cannot both exist. That is, all the vertices of G not in C are on the same side of C.

It follows that the other side of C must have an edge of G whose ends are in C. But then this edges is a chord of C, a contradiction. The assumption that the peripheral cycle C does not bound a face has led to these contradictions, so C must bound a face.

3. Let G be a graph in which the length of the shortest cycle is g.

   (a) Prove that if C is a cycle of length at most $2g-3$, then C has no chord.

   SOLUTION. *Suppose C is the cycle $(v_0, v_1, v_2, \ldots, v_{k-1}, v_0)$ of length $k \leq 2g-3$. If $0 \leq i < j \leq k-1$ are such that v_iv_j is a chord of C (in particular, v_i and v_j are not adjacent in C), then there are two cycles $(v_0, v_1, \ldots, v_i, v_j, v_{j+1}, \ldots, v_{k-1}, v_0)$ and $(v_i, v_{i+1}, v_{i+2}, \ldots, v_j, v_i)$ in G. The sum of their lengths is $(k-(j-i)+1)+(j-i+1) = k+2$. (Every edge of C is in precisely one of the two cycles and v_iv_j is in both.)*

   The sum of their lengths is $k+2 \leq (2g-3)+2 = 2g-1$ and, therefore, one of them has length $< g$, which is impossible. Thus, C has no chord.

   (b) Prove that a cycle C of length at most $2g-3$ is peripheral if and only if $G - V(C)$ is connected.

   SOLUTION. *Since C is chordless, the definition of peripheral implies C is peripheral if and only if $G - V(C)$ is connected.*

4. (a) Let $k \geq 3$ be an integer and let G be a k-connected graph. Prove that if C is a cycle of length less than k, then C is peripheral if and only if C has no chords.

   SOLUTION. *By definition of k-connected, $G - V(C)$ is connected. By definition of peripheral, C is a peripheral cycle if and only if C has no chords.*

   (b) For each $k \geq 3$, give an example of a k-connected graph that has a cycle of length k that is not peripheral.

   Let G be made up of the union of two disjoint complete graphs K_k, call them H and K, plus a k-cycle C disjoint from both H and K. Join every vertex of $H \cup K$ to every vertex of C. The result is a k-connected graph, yet $G - V(C)$ is not connected.

(c) Let $k \geq 3$ be an integer and suppose G is a k-connected graph. Let C be a cycle in G and let v be a vertex of G not in C. Prove there is a set $\mathcal{P}$ of paths from v to distinct vertices of C so that:

- $\mathcal{P} \geq \min\{k, |V(C)|\}$;
- any two paths in $\mathcal{P}$ have just v in common; and
- each path in $\mathcal{P}$ intersects C just at its end in C.

SOLUTION. *Let $t = \min\{k, |V(C)|\}$ and let $v_1, v_2, \ldots, v_t$ be distinct vertices of C. Since G is t-connected (as $t \leq k$ and G is k-connected), by Asst. 5, Q. 1, there are t paths $P_1, \ldots, P_t$ so that, for each i, P_i is a vv_i-path in G, and any two of the P_i have only v in common.*

For $i = 1, 2, \ldots, t$, let P_i' be the subpath of P_i obtained by proceeding along P_i from v and stopping at the first vertex of C encountered. Then $P_1', P_2', \ldots, P_t'$ are the desired paths.