THIRD BATCH OF HOMEWORK PROBLEMS

C1. For $n, k \in \mathbb{N}$, let $q(n,k)$ be the number of connected graphs with k edges and vertex-set $\{1,2,\ldots,n\}$; also let $Q_n(t) = \sum_{k=0}^{n(n-1)/2} q(n,k)t^k$.
(a) Explain an efficient algorithm for computing $Q_n(t)$. (Hint: if the connected components of $\mathcal{A}$-structures are $\mathcal{B}$-structures, then $A(x) = \exp(B(x))$.)
(b)* If you know `Maple` or some other computer algebra software, write some code and crank out $Q_8(t)$.
(Or do it by pencil and paper! ;-)

C2. Let $T \in \mathcal{R}_X$ be a rooted labelled tree (RLT) with vertex-set X. A vertex of T is said to be *even* or *odd* depending on whether it has an even or an odd number of children. Let $e(T)$ and $o(T)$ be the number of even or odd vertices of T, repectively. Let $\mathcal{K}_X$ be the subset of $\mathcal{R}_X$ defined by the following property: T is in $\mathcal{K}_X$ if and only if for each vertex of T, all of its children have the same parity (all are even, or all are odd). This defines a subclass $\mathcal{K}$ of $\mathcal{R}$. Derive a system of functional equations which implicitly defines the bivariate exponential generating function

$$K(x,y) := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{T \in \mathcal{K}_n} x^{e(T)} y^{o(T)} \right).$$

[Hint: first solve the similar problem in the easier case of the the class $\mathcal{R}$.]

C3. A *clique-tree* is a connected graph such that each edge is in a unique maximal complete subgraph, and each cycle is contained in a complete subgraph. Equivalently, it is a connected graph in which each block (2-connected component) is a complete subgraph. Determine the number of clique-trees with vertex-set $\{1,2,\ldots,n\}$, for each $n \in \mathbb{N}$.

C4. Refer to pages 5 to 7 of the notes on sl(2).
(a) Show that $P(V \otimes W; t) = P(V;t) \cdot P(W;t)$.
(b) Determine the multiplicities of the irreducible representations in $U(k) \otimes U(\ell)$ for all $k, \ell \in \mathbb{N}$.
(c) Determine the multiplicities of the irreducible representations in the Boolean representations $U(1)^{\otimes n}$ for all $n \in \mathbb{N}$.

C5. Let $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ be an infinite $\mathbb{Z}$-linear sum of monomials. Show that if f is invariant under all permutations in $\S_{\mathbb{P}}$ then f is invariant under all permutations in $\S_\infty$.

C6. This concerns power-sum symmetric functions.
(a) Prove that $P(-t) = E'(t)/E(t)$.
(b) Deduce that for all partitions λ, $\omega(p_\lambda) = (-1)^{|\lambda| - \ell(\lambda)} p_\lambda$.

C7. From the identity $P(-t) = E'(t)/E(t)$ for the generating functions of one-part power-sum and elementary symmetric functions, prove that for all $n \in \mathbb{N}$:

$$p_n = \det \begin{bmatrix} e_1 & 1 & 0 & 0 & \dots & 0 \\ 2e_2 & e_1 & 1 & 0 & \dots & 0 \\ 3e_3 & e_2 & e_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ (n-1)e_{n-1} & e_{n-2} & e_{n-3} & e_{n-4} & \dots & 1 \\ ne_n & e_{n-1} & e_{n-2} & e_{n-3} & \dots & e_1 \end{bmatrix}.$$

C8. For each $n \in \mathbb{N}$, let $p(n)$ be the number of partitions of n and let $c(n)$ be the number of self-conjugate partitions of n. Give a formula for the characteristic polynomial $\det(tI - \omega)$ of the linear transformation $\omega : \Lambda^n \to \Lambda^n$ in terms of $p(n)$ and $c(n)$.

C9. Prove Proposition 18 in the symmetric functions notes, the dual form of the Jacobi-Trudi Formula: $s_\lambda = \det(e_{\lambda'_i - i + j})$.

C10. Let λ and μ be partitions such that $F_\mu \subseteq F_\lambda$. A *skew tableau of shape* λ/μ is a function $T : F_\lambda \smallsetminus F_\mu \to \mathbb{P}$ which is weakly increasing from left to right along rows, and strictly increasing from top to bottom along columns. The *skew Schur function of shape* λ/μ is $s_{\lambda/\mu} := \sum_T \mathbf{x}^T$, with the sum over all skew tableau of shape λ/μ.

(a) Sketch a proof that $s_{\lambda/\mu}$ is a symmetric function. (You can be brief, but not too brief.)

(b) Derive a formula for $s_{\lambda/\mu}$ as a polynomial in the complete symmetric functions.

(c) State, without proof, a formula for $s_{\lambda/\mu}$ as a polynomial in the elementary symmetric functions.

(d) Determine the action of ω on skew Schur functions.
