C&O 442/642 Assignment 1 Due: September 21

Submit your assignment at the start of class. This is a senior level mathematics course, your solutions should be clear, concise, and logically consistent. If your solution is not essentially correct you will get no credit. You may discuss assignment solutions with another student as long as neither of you has yet written a solution; taking written notes during the discussion is considered cheating.

Problem 1: Let $G = (V, E)$ be a simple graph with no K_4-minor.

(a) Prove that, if P is a longest path in G and x is an end of P with degree at least 3, then there is a circuit C with a chord f such that $V(C) \subseteq V(P)$ and f is incident with x. (A *chord* of a cycle C is an edge, not in the cycle, with both of its ends in the cycle.)

(b) Prove that, if $f = xy$ is a chord of C, then $G - x - y$ is not connected.

(c) Prove that, if $e = uv$ is an edge of G and $|V| \geq 3$, then there is a vertex $w \notin \{u, v\}$ with degree at most 2. (Hint: Find an edge $f = xy$ distinct from e such that $G - x - y$ is not connected and apply induction.)

(d) Prove that, if V is nonempty, then G has a vertex of degree at most 2.

Problem 2:

(a) Prove that, if $G = (V, E)$ is a simple graph with at least 3 vertices and no K_5-minor, then $|E| \leq 3|V| - 6$. (Hint: Use 1(d).)

(b) Show that, for all integers $n \geq t \geq 2$, there is a simple n vertex graph with $(t-2)n - \binom{t-1}{2}$ edges and no K_t-minor. (Hint: If $G - v$ has no K_{t-1}-minor, then G has no K_t-minor.)

(c) Let G be the graph obtained from K_{10} by deleting a perfect matching. Show that G has no K_8-minor. (This beats the bound in (a) for $t = 8$ and $n = 10$.)

Problem 3: A *subdivision* of a graph H is a graph obtained by replacing the edges of G by paths of length at least 1. A graph H is a *topological minor* of a graph G if there is a subdivision of H that is a subgraph of G.

(a) Prove that a graph contains $K_{3,3}$ or K_5 as a topological minor if and only if G contains $K_{3,3}$ or K_5 as a minor. (Your proof should be self-contained.)

(b) Find an infinite sequence of graphs such that no one contains another as a topological minor. (Hint: there exists such a sequence for which none of the members contain a K_4-minor.)

(c) [CO642 only] Let $\mathcal{G}$ be a class of graphs that is closed under minors and, hence, also under topological minors. Prove that $\mathcal{G}$ has only finitely many excluded topological minors. (You may use the fact that there are only finitely many excluded minors.)

(d) What are the excluded topological minors for the class of graphs with no K_5-minor? (You do not need to justify your answer.)

Problem 4: [Bonus Problem] Let $\mathcal{G}$ be the set of all graphs G such that there exists $v \in V(G)$ where $G - v$ is a forest. Determine the excluded minors for $\mathcal{G}$. (Prove your solution.)