CO442/642 **Assignment 2** **Due: October 5**

Submit your assignment at the start of class. This is a senior level mathematics course, your solutions should be clear, concise, and logically consistent. If your solution is not essentially correct you will get no credit. You may discuss assignment solutions with another student as long as neither of you has yet written a solution; taking written notes during the discussion is considered cheating.

Problem 1: This question gives Wagner's original proof of his bound on the chromatic number of graphs with no K_n-minor.

(a) Prove that, for each integer n, a graph is the union of n bipartite subgraphs if and only if it is 2^n-colourable.

(b) Prove that, for each integer $n \geq 1$, every loopless graph with no K_n-minor is 2^{n-1}-colourable. (Hint: Consider a depth first tree from a vertex v.)

Problem 2: Prove that, if G has no path of length $k+1$, then G is k-colourable. (Hint: Fix an ordering of the vertices and the colours and colour each vertex in turn with the first available colour.)

Problem 3: A *hitting set* in a graph is a set of vertices whose deletion leaves a forest. The following problems show that every graph contains either two vertex disjoint cycles or a hitting set of size ≤ 4.

(a) Prove that, if $G = (V, E)$ is a simple graph with no K_4-minor, then either G has two vertex disjoint cycles, or there is a hitting set of size two.

(b) [CO642 only] Prove that, if H is a subdivision of K_4, X is the set of degree 3 vertices of H, and H is a subgraph of a graph G, then either G has two vertex-disjoint cycles or X is a hitting set for G.

Problem: 4 Suppose that $(T, (B_w :, w \in V(T)))$ is a tree decomposition of a graph G and H is a connected subgraph of G. Prove that, $\{w \in V(T) : B_w \cap V(H) \neq \emptyset\}$ induces a subtree of T.

Problem 5: Prove that, for non-negative integers k and l, there exists an integer h such that for each graph G either:

(1) G contains k vertex disjoint cycles each of length at least l, or

(2) there is a set $X \subseteq V(G)$ such that $|X| \leq h$ and $G - X$ has no cycle of length $\geq l$.

(Hint: You may use the Grid Theorem.)

Problem 6: Let $\mathcal{F}$ be a collection of subtrees of a tree T.

(a) Prove that, if there are no two vertex disjoint trees in $\mathcal{F}$ then there is a vertex $v \in V(T)$ that is contained in each of the trees. (Hint: For each edge $e = uv$ of T we may assume that there is a tree in $\mathcal{F}$ that avoids e. Then exactly one component of $T - e$ contains a tree in $\mathcal{F}$; use this to orient G.

(b) For any integer $k \geq 1$, prove that either there are k vertex disjoint trees in $\mathcal{F}$ or there is a set X of $< k$ vertices in T that intersects each tree in $\mathcal{F}$. (Hint: Consider the set of all edges e in T such that both components of $T - e$ contain a tree in $\mathcal{F}$; this set forms a subtree of G; consider a leaf of this subtree.)

Problem 7:[Bonus Problem] The *girth* of a graph is the length of the shortest cycle; for forests we define the girth to be infinity. Prove that, for each integer $n \geq 3$ there exists an integer g such that, every graph with girth $\geq g$ and minimum degree ≥ 3 has a K_n-minor. (Hint: Trade girth for density. Note that, if F is a forest in G and each component is a star, then the girth of G/F is at least a third the girth of G.)