CO442/642 Assignment 4 Due: November 2

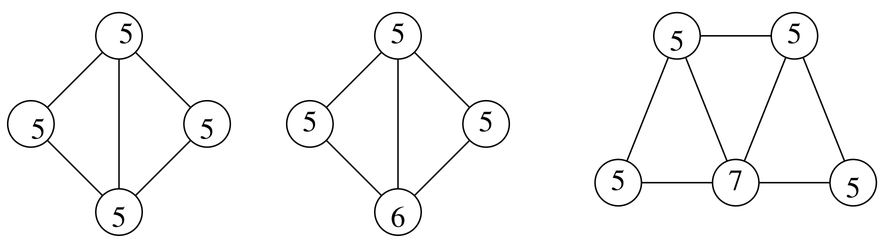
Submit your assignment at the start of class. This is a senior level mathematics course, your solutions should be clear, concise, and logically consistent. If your solution is not essentially correct you will get no credit. You may discuss assignment solutions with another student as long as neither of you has yet written a solution; taking written notes during the discussion is considered cheating.

Problem 1: A *plane quadrangulation* is a connected plane graph whose faces all have degree 4.

(a) Prove that, if $G = (V, E)$ is a plane quadrangulation, then $|E| = 2|V| - 4$.

(b) Prove that every simple plane quadrangulation with minimum degree 3 either contains an edge uv where u has degree 3 and v has degree 3 or 4, or contains a degree 5 vertex with 4 neighbours of degree 3.

Problem 2: Prove that, if G is a simple plane triangulation with minumum degree 5, and G does not contain any of the configurations below, then G has a vertex v with degree 5 or 6 such that each neighbour of v has degree < 30.



Problem 3:

(a) Let $C = (V, E)$ be a cycle of even length and, for each $v \in V$, let $S(v)$ be a set containing two distinct colours. Prove that there is a colouring of G such that each vertex obtains one of its two preassigned colours. (Here, as usual, adjacent vertices cannot get the same colour.)

(b) Prove that no minumum counterexample to the 4-colour theorem contains a vertex of degree 6 whose neighbours all have degree 6.

Problem 4:

(a) A *tournament* is an orientation of a simple complete graph. A tournament is *acyclic* if it has no directed cycle. Use Ramsey's Theorem to prove that, for each nonnegative integer n there is an integer T, so that every tournament on T or more vertices contains an acyclic tournament on n vertices.

(b) Reprove (a) without using Ramsey's Theorem.

(c) Use Ramsey's Theorem to prove that, for each nonnegative integer n there is an integer S such that every sequence of $m \geq S$ distinct integers contains a subsequence of length n that is either monotinically increasing or monotonically decreasing.

(d) Reprove (b) without using Ramsey's Theorem.

(e) Prove Ramsey's Theorem using the results in (a) and (c).

Problem 5: [CO642 Only] In this question by an *edge-colouring* we mean a colouring of the edges with no other conditions. An edge-colouring of K_n is *speckled* if no two edges have the same colour. An edge-colouring of K_n is *monochromatic* if all edges have the same colour. A *rainbow* colouring of K_n is an edge-colouring such that, there is an ordering $(1, 2, \dots, n)$ of the vertices of K_n and distinct colours $(c_1, c_2, \dots, c_{n-1})$ so that, for each $0 < i < j \leq n$, the edge ij gets colour c_i. Prove that, for each positive integer n there exists an integer C such that, every edge-colouring of a complete graph on at least C vertices contains a complete subgraph with n vertices that is either monochromatic, speckled, or rainbow coloured.

Problem 6: [Bonus Problem] Prove that, if G is a simple planar graph with minimum degree 5, then there exist two distinct degree 5 vertices u and v in G and a (u, v)-path whose vertices all have degree at most 6.

Problem 7: [Bonus Problem] Is it true that, if G is a simple planar graph with minimum degree 5, and H is the induced subgraph containing the degree 5 and 6 vertices of G, then there is a component H_0 of H containing at least 12 vertices that have degree 5 in G? (I don't know the answer, but I have not tried to prove it yet — Jim.)