CO442/642 **Assignment 6** **Due: November 28**

Submit your assignment at the start of class. This is a senior level mathematics course, your solutions should be clear, concise, and logically consistent. If your solution is not essentially correct you will get no credit. You may discuss assignment solutions with another student as long as neither of you has yet written a solution; taking written notes during the discussion is considered cheating.

Problem 1: Prove that, for any integer $k \geq 1$ and any $0 < p < 1$, a graph $G \in \mathcal{G}_{n,p}$ is asymptotically almost surely k-connected. (Hint: We proved the result for $k = 1$ in class.)

Problem 2: In class we proved, for any simple graph $G = (V, E)$, that

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{\deg(v) + 1}.$$

(a) Prove that, equality is attained if and only if G is a disjoint union of cliques. (Hint: equality is only attained when for each ordering of the vertices, the algorithm finds a maximum stable set.)

(b) Use the above to prove Turán's Theorem. (Hint: Show that, among graphs with n vertices and m edges, the right hand side is minimized when no two vertex degrees differ by two or more.)

Problem 3: Let $l \geq 1$ be an integer and, for each integer $n \geq 1$, let $p(n) = n^{\frac{1}{2l}-1}$.

(a) Let $0 < \delta < 1$. For $G \in \mathcal{G}_{n,p(n)}$, let x denote the number of cycles of length at most l. Prove that, $x \leq \delta n$ asymptotically almost surely.

(b) Let $k \geq 1$ be a integer. Prove that, for $G \in \mathcal{G}_{n,p(n)}$, we have $\alpha(G) < \frac{n}{2k}$ asymptotically almost surely.

(c) Let $0 < \delta < 1$. Prove that, $G \in \mathcal{G}_{n,p(n)}$ asymptotically almost surely has a subgraph H with girth $\geq l$, chromatic number $\geq k$, and $|V(H)| \geq (1 - \delta)n$.

Problem 4: Let Γ be a finite abelian group, let T be a spanning tree of an oriented graph $\vec{G}$ and let $f' : E(\vec{G}) - E(T) \to \Gamma$. Prove that there exists a unique Γ-flow f such that $f(e) = f'(e)$ for all $e \in E(\vec{G}) - E(T)$.

Problem 5: Prove that, if a graph has a 2-decomposition, then it has a nowhere zero $\mathbf{Z}_2 \times \mathbf{Z}_3$-flow.

Problem 6: [CO642 Only] Let v_1 and v_2 be distinct vertices in a graph G' and let G be the graph obtained by identifying v_1 and v_2 into a single vertex v. We say that G' is obtained from G by *splitting* v.

(a) Let v be a vertex of degree at least 4 in a bridgeless graph G. Prove that there there is a bridgeless graph G' that can be obtained from G by splitting v into two vertices of degree ≥ 2.

(b) Prove that, if the 5-Flow Conjecture fails, then there exists a counterexample that is 3-regular, simple, and 3-connected.