C&O 444/644

Assignment 1

1. Show that the cube is a Cayley graph for $\mathbb{Z}_2^3$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$. [What generalizations hold for the n-cube?]

2. Let $X(G, C)$ be a Cayley graph for the group G and suppose S is an independent set of vertices in X. If a and b are adjacent vertices in X, prove that $S^{-1}a \cap S^{-1}b = \emptyset$. Using this prove that, if X is a Cayley graph, then $\alpha(X)\omega(X) \leq |V(X)|$. [Provide an example of a connected Cayley graph for an abelian group, neither 2-colourable nor complete, where equality holds.]

3. Let L an $n \times n$ Latin square with elements. Define $X(L)$ to be the graph with vertex set consisting of all n^2 triples $(i, j, L_{i,j})$, where two triples are adjacent if they agree on exactly one coordinate. If L is the multiplication table of a group G, prove that $X(L)$ is a Cayley graph.

4. If the graph X is a Cayley graph both for $\mathbb{Z}_2^{2d}$ and $\mathbb{Z}_4^d$, prove that 2^d divides the order of a vertex stabilizer. [Remark: the only proof of this that I know uses results about Sylow 2-subgroups.]

5. Prove that any automorphism of a tree either fixes a vertex or an edge.

6. Let $X = X(G, C)$ be a Cayley graph for G. Characterize the pairs (G, C) such that X is bipartite.

7. The vertices of the Petersen graph can be taken to be the unordered pairs of elements from $\{0, 1, 2, 3, 4\}$, where two pairs are adjacent if the they are disjoint. This graph has five independent sets of size four. Construct the Clebsch graph by first taking five new vertices $v_0, \ldots, v_4$ and joining v_i to the four vertices in the i-th independent set. Then take a sixth vertex w and join it to each of the vertices $v_0, \ldots, v_4$.

   Prove that the Clebsch graph is isomorphic to the Cayley graph for $\mathbb{Z}_2^4$ with connection set $\{e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4\}$. (Here $e_1, \ldots, e_4$ is the standard basis for $\mathbb{Z}_2^4$, viewed as a 4-dimensional vector space over $GF(2)$.)

8. Prove that any connected triangle-free cubelike graph is a spanning subgraph of a triangle-free cubelike graph with diameter two.

9.

10. Let Ω denote the set of partitions of $\{1,\dots,9\}$ into three disjoint triples. The symmetric group $\mathrm{Sym}(9)$ acts on Ω.

    (a) Show that $\mathrm{Sym}(9)$ acts transitively on Ω. (You may be brief.)

    (b) Compute the size of a stabilizer of a partition, and so determine $|\Omega|$.

    (c) Determine the number of orbitals of $\mathrm{Sym}(9)$, and determine which orbitals are graphs.

    (d) [bonus] Show that the subgroup $\mathrm{Sym}(8)$ of $\mathrm{Sym}(9)$ acts transitively on Ω. (Your proof should **not** require extensive computation.)

11. If X and Y are graphs, their *Cartesian product* $X \,\square\, Y$ is defined as follows. The vertices of $X \,\square\, Y$ are the elements of $V(X) \times V(Y)$, and $(x_1, y_1) \sim (x_2, y_2)$ if and only if either

    (a) $x_1 = x_2$ and $y_1 \sim y_2$, or

    (b) $x_1 \sim x_2$ and $y_1 = y_2$.

    Suppose M is a $d \times n$ matrix over $\mathbb{Z}_2$ of the partitioned form

    $$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

    Show that the cubelike graph $X(M)$ is isomorphic to $X(M_1) \,\square\, X(M_2)$.

12. If H and K are subsets of G, then HK denotes the subset

    $$\{hk : h \in H, k \in K\}$$

    If $H, K \leq G$ and $g \in G$, then HgK is called a *double coset*. (It is a disjoint union of right cosets of H and of left cosets of K.) If G is a transitive group of permutations of V and $H \leq G$, show that the orbits of H correspond to the double cosets of the form $G_x gH$, where $g \in G$ and $x \in V$. Show further that the orbital that corresponds to the double coset $G_x g G_x$ is self-paired if and only if $g^{-1} \in G_x g G_x$.

13. Let r be fixed, and suppose that for each pair of distinct vertices u and v in X, there is an r-coloring of X where u and v get different colors. Show that X is a subgraph of a product of some number of copies of K_r, i.e., a subgraph of K_r^m or some m.

14. Let X be the graph with the elements of $GF(16)$ as its vertices, with a and b adjacent if and only $a - b$ is a non-zero cube. Prove that X is isomorphic to the Clebsch graph. Prove that $\chi(X) = 4$.

15. Prove that the edge set of K_{16} can be partitioned into three edge-disjoint copies of the Clebsch graph.

16. Determine the core of $L(K_n)$ for all n.

17. Show that an automorphism of the d-cube that fixes a vertex and each of its neighbors is the identity.

18. Prove or disprove: any automorphism of the Kneser graph $K_{2k+1:k}$ that fixes each neighbor of a vertex is the identity.

19. Prove or disprove: if $v \geq 2k + 1$, the Kneser graph $K_{v:k}$ is distance transitive.

20. If X is a graph, let $\mathcal{N}(X)$ denote the multiset of neighborhoods of X. (Here a neighborhood is just a set of vertices.) If X and Y are graphs with the same vertex set, show that $X \times K_2 \cong Y \times K_2$ if and only $\mathcal{N}(X) \cong \mathcal{N}(Y)$.

21. Prove that a cubelike graph that contains a triangle contains a copy of K_4.

22. If X is a Cayley graph for an abelian group, prove that $X \,\square\, X \to X$.

23. Let P'_n be the graph formed from the path with vertex set $\{0, \ldots, n-1\}$ by putting a loop on 0. Construct the graph $\Delta_n(X)$ by adding a new vertex to $P'_n \times X$ and joining it to each vertex in the set

$$\{(n-1, u) : u \in V(X)\}.$$

Show that there is a proper r-coloring of $\Delta_n(X)$ if and only if there is a walk of length $n-1$ in K_r^X whose first vertex is a homomorphism and whose last vertex is a constant function.

24. Show that the only primitive permutation group that contains a transposition is the symmetric group.

25. If X is a Cayley graph for an abelian group, prove that $\mathrm{Aut}(X)$ is generously transitive.

26. Let X be a 4-regular graph such that the neighbours of a vertex induce a copy of $2K_2$. Prove that the stabilizer of a vertex is a 2-group.

27. Show that the automorphism group of the line graph of the Petersen graph is distance-transitive and imprimitive.

28. Let X be a graph with n vertices. Construct a graph $\mathbb{Z}_2(X)$ as follows. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{Z}_2^n$. The vertices of $\mathbb{Z}_2(X)$ are the vectors in $\mathbb{Z}_2^n$; vertices u and v are adjacent if $u + v$ is equal to $e_i + e_j$ where $ij \in E(X)$. How many components does $\mathbb{Z}_2(X)$ have?

29. Prove that any graph is an induced subgraph of a cubelike graph.