C&O 444/644

Assignment 2

1. If X is a connected arc-transitive graph, prove that any proper block for $\mathrm{Aut}(X)$ induces a coclique. [If X is 2-arc transitive and not bipartite, prove that any two vertices in a proper block are at distance at least three in X.]

2. Show that a group G acts 2-arc-transitively on a connected graph X with minimu valency two if and only if it acts transitively, and for any vertex u the stabilizer G_u acts 2-transitively on the set of neighbors of u.

3. If M and N are two $m \times n$ matrices over a finite field $\mathbb{F}$, then $\mathrm{rk}(M) = \mathrm{rk}(N)$ is and only if there are invertible matrices A and B such that $N = AMB^T$. (Believe me.) Define a graph whose vertices are the $m \times n$ matrices over $\mathbb{F}$, where two matrices are adjacent if and only if their difference has rank one. Prove that this graph is a Cayley graph and that it is distance transitive.

4. Prove that a graph that is triangle-free and distance transitive is 2-arc transitive.

5. Prove that the edge connectivity of an edge-transitive graph is equal to its minimum valency (or present a counterexample).

6. If X is a graph, let $X[m]$ denote the graph we get by replacing each vertex of X by a coclique of size m and adding all possible edges between the cocliques corresponding to adjacent vertices. (So $K_2[m] \cong K_{m,m}$.) If X is vertex transitive and $m = |\mathrm{Aut}(X)_1|$, prove that $X[m]$ is a Cayley graph for $\mathrm{Aut}(X)$.

7. A *circular ladder graph* is a Cartesian product $K_2 \,\square\, C_n$. Show that any circular ladder graph is a Cayley graph. If n is odd, show that $K_2 \,\square\, K_n$ is a circulant.

8. Determine the core of a circular ladder graph.

9. The *Möbius ladder* $M(2n)$ is constructed from the product $K_2 \,\square\, P_n$ by addin the edges $((0,1),(1,n))$ and $(1,1),(0,n)$. Show that it is a circulant.

10. Determine the cores of the Möbius ladders.

11. If X is vertex transitive, prove that the permutation rank of $\mathrm{Aut}(X^\bullet)$ is less than or equal to the permutation rank of $\mathrm{Aut}(X)$.

12. Suppose Y and Z are graphs and f is a homomorphism from Y to Z. Show that if Z is cubelike, there is a homomorphism $\hat{f}$ from $\mathbb{Z}_2(Y)$ to Z which is $\mathbb{Z}_2$-linear.

13. The folded d-cube is the graph on 2^{d-1} vertices that we get by identifying antipodal vertices in the d-cube. Prove that the folded $(d+1)$-cube is isomorphic to the Cayley graph $X(\mathbb{Z}_2^d, \{e_1, \ldots, e_d, e_1 + \cdots + e_d\})$.

14. Let B be the incidence matrix of a graph, viewed as a matrix over $\mathbb{Z}_2$. Show that there are binary vectors a and b such that for each r at least one of $(a^T B)_r$ and $(b^T B)_r$ is not zero, if and only if X is 4-colorable.