PMATH 145 Assignment 1

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due September 29th by 12:30PM

Problem 1. The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2}$$

for all $n \geq 2$, so that $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, *etc.* (WARNING: The textbook defines the sequence differently, so that all terms are shifted by 1. Ignore that.)

(a) Prove that $F_n < 2^n$ for all $n \in \mathbb{N}$. (Induction might be helpful.)

(b) Prove that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

for all $n \in \mathbb{N}$.

(c) Prove that

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2$$

for all $n \in \mathbb{N}$.

(d) Let $\tau = (1 + \sqrt{5})/2$, the golden ratio. Prove that

$$F_n = \frac{\tau^n - (-1/\tau)^n}{\sqrt{5}}$$

for all $n \in \mathbb{N}$. (It might be useful to note that $\tau^2 - \tau - 1 = 0$.)

(e) Prove that

$$F_{n+m} = F_m F_{n+1} + F_{m-1} F_n$$

for all $n, m \in \mathbb{N}$.

Note: all of these properties hold for $n = 0$, too. I just asked you to prove them for $n \in \mathbb{N}$ for convenience.

Problem 2. Recall that $\sqrt{2}$ is irrational, and define

$$\mathbb{Q}(\sqrt{2}) = \left\{ a + b\sqrt{2} : a, b \in \mathbb{Q} \right\}.$$

(a) Show that if $a + b\sqrt{2} = c + d\sqrt{2}$, for some $a, b, c, d \in \mathbb{Q}$, then $a = c$ and $b = d$. This shows that the representation of an element of $\mathbb{Q}(\sqrt{2})$ in the form $a + b\sqrt{2}$ is unique.

(b) Let $x, y \in \mathbb{Q}(\sqrt{2})$. Show that 0, 1, $x + y$, xy, and $-x$ are in $\mathbb{Q}(\sqrt{2})$. This shows that $\mathbb{Q}(\sqrt{2})$ is a *subring* of $\mathbb{R}$. (The definition of a subring is that it's a subset of a ring which is closed under addition, multiplication, and taking additive inverses, and contains 0 and 1. A subring automatically satisfies the criteria for being a ring.)

(c) Show that $\mathbb{Q}(\sqrt{2})$ is a field. That is, show that if $x \neq 0$, then there is an element $x^{-1} \in \mathbb{Q}(\sqrt{2})$ such that $x \cdot x^{-1} = 1$.

Problem 3. Let

$$\mathbb{Z}[\sqrt{2}] = \left\{ a + b\sqrt{2} : a, b \in \mathbb{Z} \right\},$$

which is clearly a subset of $\mathbb{Q}(\sqrt{2})$. For $x = a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$, we define the *conjugate* of x over $\mathbb{Z}$ to be $\overline{x} = a - b\sqrt{2}$, and define the *norm* by $N(x) = x \cdot \overline{x}$.

(a) Prove that $N(x) \in \mathbb{Z}$, for all $x \in \mathbb{Z}[\sqrt{2}]$.

(b) Prove that $N(xy) = N(x)N(y)$, for all $x, y \in \mathbb{Z}[\sqrt{2}]$.

(c) An element x of a ring is called a *unit* if and only if there is an element x^{-1} in the ring such that $x \cdot x^{-1} = 1$ (so, another way of defining a field is by saying that it's a commutative ring in which everything except 0 is a unit). Note that this definition depends on the ring: 2 is a unit in $\mathbb{Q}$, but isn't a unit in $\mathbb{Z}$.

Show that $17 + 12\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ is a unit.

(d) Prove that $x \in \mathbb{Z}[\sqrt{2}]$ is a unit if and only if $N(x) = \pm 1$.

(e) Prove that there are infinitely many units in $\mathbb{Z}[\sqrt{2}]$.

Problem 4. Recall that if R is a commutative ring, and $a, b \in R$, then the notation $a \mid b$ means that there exists a $c \in R$ such that $b = ac$.

(a) Let $a, b, c, d, e \in \mathbb{Z}$, and suppose that $a \mid b$ and $a \mid c$. Show that $a \mid (db + ce)$.

(b) Let R be a field. Prove that for $a, b \in R$, we have $a \mid b$ whenever $a \neq 0$, and that $0 \mid b$ if and only if $b = 0$.

Problem 5. Prove that for every $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that none of the numbers

$$n, n + 1, n + 2, ..., n + m$$

is prime.