PMATH 145 Assignment 5

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Due November 17 at 12:30PM

Problem 1. Recall that $\sigma(n) = \sum_{d|n} d$, a multiplicative function, and that n is called *perfect* if and only if $\sigma(n) = 2n$.

i. If p and q are odd primes, show that $n = pq$ is not perfect.

ii. Show that if $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ is an odd perfect number (with the p_i distinct primes), then exactly one exponent is odd, and the rest are even.

iii. Show that if $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ is an odd perfect number and e_1 is the odd exponent, then in fact $e_1 \equiv 1 \pmod 4$ and $p_1 \equiv 1 \pmod 4$.

Problem 2. The function $f(n) = \sum_{d|n} \sigma(n)$ is multiplicative. Find a formula for $f(n)$ when

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}.$$

Problem 3. In coding theory, it is useful to be able to count the number of binary sequences of length n which are not made by repeated a sequence of shorter length. For example, we'd like to be able to distinguish between the two sequences of length 4:

$$0111 \qquad \text{versus} \qquad 0101.$$

The second sequence is made by repeating the shorter sequence 01 twice, but the first can't be made by repeating any shorter sequence. Let $S(n)$ be the number of sequences of length n which are not made by repeating shorter sequences.

i. Show that $\sum_{d|n} S(n) = 2^n$.

ii. Find $S(3)$, $S(10)$, and $S(100)$.

Problem 4. Give an example of a field F and a polynomial $f(x) \in F[x]$ which is not the zero polynomial (i.e., f has at least one non-zero coefficient) but $f(c) = 0$ for all $c \in F$.