PMATH 145 Assignment 6

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Due December 3rd at 12:30PM

Problem 1. Compute $\gcd(x^5 - x^2 + x^3 - 1, x^5 - x^3 - 2x)$, and write the result in the form

$$s(x)(x^5 - x^2 + x^3 - 1) + t(x)(x^5 - x^3 - 2x).$$

Problem 2. Show that the infinite series

$$\sum_{m=1}^{\infty} 2^{-2^m}$$

converges to a transcendental number.

Problem 3. In 1909, Axel Thue improved on Liouville's Theorem, proving that if $a \in \mathbb{R}$ is the root of an irreducible polynomial of degree $d \geq 2$, with coefficients in $\mathbb{Q}$, then there is a $\delta > 0$ such that

$$\left| a - \frac{p}{q} \right| > \frac{\delta}{q^{2\sqrt{d}}}$$

for all $p/q \in \mathbb{Q}$. (This is an improvement because $2\sqrt{d} < d$ for $d \geq 5$.) Use this to prove that if

$$f(X, Y) = a_d X^d + a_{d-1} X^{d-1} Y + a_{d-2} X^{d-2} Y^2 + \cdots + a_1 X Y^{d-1} + a_0 Y^d,$$

where $a_i \in \mathbb{Z}$, $d \geq 5$, and $f(x, 1) \in \mathbb{Z}[x]$ is irreducible, then the equation $f(X, Y) = 1$ has only finitely many solutions $X, Y \in \mathbb{Z}$. (Note: the expression $f(X, Y)$ has the property that, in each term, the exponent of X and the exponent of Y sum to d. It might be useful to use the fact that $f(p, q) = q^d f(p/q, 1)$.)

Problem 4. Show directly that the commutative ring $\mathbb{Z}_5[x]/(x^2 - 2)$ is a field, by exhibiting a multiplicative inverse of every non-zero element.

Problem 5. Let R be a finite commutative ring, and let $U \subseteq R$ be the set of all elements that have multiplicative inverses. Prove that for any $a \in U$, we have

$$a^{\#U} = 1.$$

(This generalizes Euler's Theorem.)