Math 146 Assignment 10
Due 1:00 pm on Wednesday, Mar 30, 2011

1. In the previous assignment we came across the following result
For the unit (normal) vector

$$\hat{x} = \frac{1}{\sqrt{6}}(1, -1, 2),$$

the orthogonal projection $Proj_{\hat{x}}$ has matrix representation (with respect to the standard basis in $\mathbb{R}^3$) given by

$$\begin{bmatrix} \frac{1}{6} & \frac{-1}{6} & \frac{1}{3} \\ \frac{-1}{6} & \frac{1}{6} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix}.$$

Treat $\hat{x}$ as the column matrix:

$$\hat{x} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

(i) Multiplying the matrix $\hat{x}$ by its transpose on the right side, verify that we indeed arrive at the above matrix representation of $Proj_{\hat{x}}$.

(ii) Prove the following generalization.

Proposition. If $\hat{x}$ is any unit vector in $\mathbb{R}^3$. Then the orthogonal projection $Proj_{\hat{x}}$ is represented by the matrix $A = \hat{x}\hat{x}^t$.

[Hint: Observe that dot product in $\mathbb{R}^n$ connects to matrix multiplication via

$$x \cdot y = y^t x, \quad x \cdot y = x \cdot y \quad (symmetry), \quad y^t x = x^t y.$$

Check that each vector y on the line spanned by $\hat{x}$ is mapped by L_A to y, and that each vector y orthogonal to $\hat{x}$ is mapped by L_A to 0. The two features will be enough to conclude that A is the matrix representation of the orthogonal projection $Proj_{\hat{x}}$.]

(iii) Is the above true in $\mathbb{R}^n$ $(n > 3)$. Just say yes or no.

2. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{bmatrix}.$$

Evaluate $\det(A - \lambda I_n)$, the *characteristic polynomial* of A.

3. Prove that if $M \in M_{n\times n}(F)$ has blocked form

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A and C are square matrices, then $\det(M) = \det(A)\det(C)$.

4. The *trace* of a square matrix $A = [a_{ij}]$ is the sum of its diagonal entries a_{ii}. (i) Prove that the trace function satisfies $trace(AB) = trace(BA)$ for all $n \times n$ matrices A and B. (ii) Is it true that $trace(AB) = trace(A)trace(B)$? (iii) Show that similar matrices have same trace. (iv) Exhibit an example demonstrating that two $n \times n$ matrices having the same determinant does not imply that they are similar.

5. Let A be an $n \times n$ square matrix over F. Roots of its characteristic polynomial $\det(A - \lambda I_n)$ in F are called the *eigenvalues* of A.

For a rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, what are its eigenvalues ?