Math 146 Assignment 3
Due 1:00 pm on Wednesday, February 2, 2011

Please submit your assignments in Drop Box #10, Slot#12, outside the tutorial center on the fourth floor of the MC building.

1. Let $f : X \to Y$ and $g : Y \to Z$ be functions. The *composition* , $g \circ f$, is the map from X to Z defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$. (i) Show that if f and g are injective then $g \circ f$ is injective. (ii) Show that if f and g are surjective then $g \circ f$ is surjective.

Definition Let X and Y be sets. We say that X has smaller cardinality than Y if there exists an injective function $f : X \to Y$. In that case we write $|X| \leq |Y|$. The first part of the previous question can be rephrased as the transitivity : If $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$.

2. Let V be a vector space over $\mathbb{Z}_p$ (prime p). (i) Let v_1 be a fixed vector of V. Show that the following two statements are equivalent.

(a) $\{v_1\}$ is linearly independent. (b) The function $f : \mathbb{Z}_p \to V$ defined by $f(\lambda) = \lambda v_1$ is injective.

(ii) Let v_2, v_2 be two fixed distinct vectors of V. Show that the following two statements are equivalent.

(a) $\{v_1, v_2\}$ is linearly independent. (b) The function $f : \mathbb{Z}_p^2 \to V$ defined by $f(\lambda_1, \lambda_2) = \lambda_1 v_1 + \lambda_2 v_2$ is injective.

(iii) How does the above generalize to a set of n distinct vectors for $n = 3, 4, ...$? Give the statement without a formal proof.

3. Show that if V is a set with 8 elements, then no matter how we define addition on V and multiplication by scalars from Z_3, V cannot be made a vector space over Z_3.

4. Let $W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 \,|\, a_1 - a_3 - a_4 = 0\}$ and $W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 \,|\, a_1 + 6a_3 - a_4 + 2a_5 = 0 \text{ and } a_2 = a_3\}$. Find a basis for $W_1 \cap W_2$.

5. (i) Exhibit an example of a subset of $\mathbb{R}^3$ which is linearly independent but does not span $\mathbb{R}^3$. (ii) Exhibit an example of a subset of $\mathbb{R}^3$ which spans $\mathbb{R}^3$ and is not linearly independent. No proof is necessary.