Math 146 Assignment 7
Due 1:00 pm on Wednesday, Mar 9, 2011

Please submit your assignments in Drop Box #10, Slot#12, outside the tutorial center on the fourth floor of the MC building.

1.(i) Let V be a finite dimensional vector space. Let T_1 and T_2 be linear operators on V. Show that if T_1T_2 (composition) is bijective, then T_1 and T_2 are bijective. [Hint: the rank and nullity theorem.]

(ii) Give a counter example to show that the above statement is not true for some infinite dimensional space V.

2. Consider the vector space $P_2(\mathbb{Z}_3)$ and the linear operator L given by $L(p(x)) = p(x+1)$ (as an illustration, $L(x^2+1) = (x+1)^2+1 = x^2+2x+1+1 = x^2+2x+2$.)

Let α be the (standard) ordered basis $\{1, x, x^2\}$ for $P_2(\mathbb{Z}_3)$.
(i) Find $[L]_\alpha$.
(ii) Find $([L]_\alpha)^3$.

3. (i) Use a sequence of elementary row operations to bring the matrix

$$\begin{bmatrix} 2 & 8 & 1 & 0 & 7 & 0 \\ -3 & -12 & 0 & 2 & 2 & 0 \\ 5 & 20 & -2 & -1 & 0 & 0 \end{bmatrix}$$

to its reduced row echelon form. Indicate clearly all your steps. Use the markings $R_i <-> R_j$ to indicate the interchange of the i-th and the j-th row, $\lambda R_i \to R_i$ to indicate the multiplication of the i-th row by the non-zero constant λ and $\lambda R_i + R_j \to R_j$ to indicate that λR_i is added to the j-th row.

(ii) Based on the above result, write out the general solution for the system of equations

$$\begin{aligned} 2x_1 + 8x_2 + x_3 + 7x_5 &= 0 \\ -3x_1 - 12x_2 + 2x_4 + 2x_5 &= 0 \\ 5x_1 + 20x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

Use the variables not corresponding to a leading 1 as the free variables.

4. The transpose of an $m \times n$ matrix A, denoted A^t, is the $n \times m$ matrix obtained from A by interchanging its row with its columns. That is, the (i,j) entry of A is the (j,i) entry of A^t. (See page 17, textbook). A square matrix A is symmetric if $A^t = A$. A square matrix A is skew symmetric if $A^t = -A$.

(i) Show that the vector space $M_{3\times 3}(\mathbb{R})$ is the *direct sum* of W_1, the subspace of all symmetric matrices and W_2, the subspace of all skew symmetric matrices.

(ii) Let t denote the transpose operator on $M_{3\times 3}(\mathbb{R})$, i.e., it maps A to A^t. Show that W_1 and W_2 are t-invariant subspaces.

(iii) Find the dimensions of W_1 and W_2.

(iv) Let α and β be ordered bases for W_1 and W_2 respectively and let γ be their union, with vectors in β following those in α (this is a description of how the union is ordered). Find the matrix $[t]_\gamma$.