Math 146 Assignment 9
Due 1:00 pm on Wednesday, Mar 23, 2011

Please submit your assignments in Drop Box #10, Slot#12, outside the tutorial center on the fourth floor of the MC building.

Preamble: The *dot product* between two vectors $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$ of $\mathbb{R}^n$ is defined by

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

The Euclidean *norm (length)* of a vector x of $\mathbb{R}^n$ is defined by

$$||x|| = \sqrt{||x||} = \sqrt{(\sum_{i=1}^{n} x_i^2)}.$$

The Euclidean *distance* between two points x and y of $\mathbb{R}^n$ is defined by

$$d(x, y) = ||x - y||.$$

x and y are (mutually) *orthogonal* if $x \cdot y = 0$.

A *unit* (or *normal*) vector is a vector whose norm is equal to 1. We shall denote such a vector by $\hat{x}$.

For a non-zero vector x, $\frac{1}{||x||}x$ is a unit vector. Such adjustment-in-length action is called "normalize x".

The *orthogonal projection* of $\mathbb{R}^n$ onto the line spanned by a unit vector $\hat{x}$ is the map $Proj_{\hat{x}}$ given by the formula

$$Proj_{\hat{x}}(y) = (y \cdot \hat{x})\hat{x} \quad \forall y \in \mathbb{R}^n.$$

A note on Convention. Points in $\mathbb{R}^n$ are denoted by $(x_1, ..., x_n)$ and by $\begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix}$. We dynamically switch between the two forms based on the context it is being used.

1. Consider the Euclidean plane $\mathbb{R}^2$. (i) Show that rotations are *dot product preserving.* That is, for $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ show that

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

for all x, y. (ii) Show that a rotation is norm preserving, i.e., $||Ax|| = ||x||$ for all x.

2. (i) Find a unit vector $\hat{x}$ on the line spanned by $(1, -1, 2)$ and obtain the matrix representation A of the linear operator $Proj_{\hat{x}}$ with respect to the standard basis for $\mathbb{R}^3$ (for both the domain and codomain). (ii) Is it true that $A^2 = A$?

3. The *column space* of a matrix is the span of all its columns. Is it true that the column space of a matrix equals the column space of its RREF?

4. The *row space* of a matrix is the span of all its rows. Is it true that the row space of a matrix equals the row space of its RREF?

5. The *row rank* of a matrix is the dimension of its row space. Is the row rank of a matrix equal to its rank? [Remark: Column space of a matrix A is equal to the range space of L_A, thus rank of A, which is defined as the rank of L_A, is the dimension of the column space of A. One may say that rank of A in this definition is the column rank of A.]

6. Evaluate the determinant of the complex matrix

$$A = \begin{bmatrix} \lambda & 2 & 0 \\ -4 & \lambda - i & 0 \\ 0 & 0 & \lambda - 2 \end{bmatrix}$$

and find all values of λ for which A is invertible.