Math 147 Assignment 1 - due Friday, September 24, 2010 - 11:30 a.m.

1. Prove $|x^3 - 2x + 1| \leqslant \frac{5}{4}|x-1|$ if $-1 < x < 0$.

2. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(1) = 5, f(2) = 13$ and for $n \geqslant 3$, $f(n) = 2f(n-2) + f(n-1)$.

   Prove $f(n) = 3 \cdot 2^n + (-1)^n$ for all $n \in \mathbb{N}$.

3. Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for each $n \in \mathbb{N}$. (You may use the fact that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$).

4. Suppose $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ for all $n \in \mathbb{N}$. Show by induction that $x_n < x_{n+1} < 2$ for all $n \in \mathbb{N}$.

5. Suppose x, y, a, b are real numbers and $\epsilon > 0$. Suppose $|x-a| < \epsilon$ and $|y-b| < \epsilon$. Prove $|xy - ab| < \epsilon(|a| + |b|) + \epsilon^2$.

6. If $m, n \in \mathbb{Z}$, $n \neq 0$ show that $|\sqrt{3} - \frac{m}{n}| \geqslant \frac{1}{5n^2}$.
   (Hint: Rationalize the numerator and use the irrationality of $\sqrt{3}$.)

7. Given any real numbers x, y, with $x < y$, prove there is an irrational number z with $x < z < y$.
   (You may use (without proof) the fact that if $r > 0$ then there is some $n \in \mathbb{N}$ with $\frac{1}{n} < r$.)

Bonus: The arithmetic and geometric means of $a_1, \ldots, a_n \geqslant 0$ are given by

$$A_n = \frac{a_1 + \cdots + a_n}{n} \text{ and } G_n = \sqrt[n]{a_1 \ldots a_n}$$

Suppose $a_1 \leqslant A_n$ and $a_2 \geqslant A_n$. Put $\bar{a}_1 = A_n$ and $\bar{a}_2 = a_1 + a_2 - \bar{a}_1$. Show that $\bar{a}_1\bar{a}_2 \geqslant a_1 a_2$

Use this process, repeated enough times, and induction to prove $G_n \leqslant A_n$ for all $n \in \mathbb{N}$.