Math 147 Assignment 2 - Due Friday October 1, 2010

1. (a) Find the LUB and GLB (if it exists) for the following sets. Give a brief explanation.

   (i) $\{2^{-n} : n \in \mathbb{Z}\}$

   (ii) $\{(-1)^n(1 - \frac{1}{n}) : n \in \mathbb{N}\}$

   (iii) $\{\frac{1}{|x|+1} \ : \ x \notin \mathbb{Q}\}$

   (b) Suppose $E \subseteq \mathbb{R}$ is a bounded set. Assume x is an upper bound for E and that $x \in E$. Prove x is the LUB of E.

   (c) Show that the least upper bound of the set $\{z \in \mathbb{Q} : z^2 < 2\}$ is a real number $w > 0$ with the property that $w^2 = 2$.

2. Evaluate the following limits. Carefully justify your answer.

   (i) $\lim_{n\to\infty} \frac{(-1)^n n}{2^n}$

   (ii) $\lim_{n\to\infty} \frac{n^2+1}{n^2+2}$

   (iii) $\lim_{n\to\infty} (2^n+3^n)^{1/n}$

3. (a) Suppose $a_n \in \mathbb{Z}$ for all n and the sequence (a_n) converges. What can you say about this sequence?

   (b) Prove that for every $x \in \mathbb{R}$ there is a sequence (x_n) with $x_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$.

4. A real number x is called an accumulation point of a set $A \subseteq \mathbb{R}$ if for every $\varepsilon > 0$ there is some $a \in A$ with $a \neq x$ and $|x - a| < \varepsilon$.

   (a) Find all the accumulation points of each of the following sets. Justify your answer.

   (i) $\{1/n : n \in \mathbb{N}\}$

   (ii) $\{1/n + 1/m : m, n \in \mathbb{N}\}$

   (iii) $\{\sin(1/x) : 0 < x < 1\}$

   (b) Suppose $E \subseteq \mathbb{R}$ is non-empty and bounded above. Let $z = \text{LUB}(E)$. Prove that if $z \notin E$, then z is an accumulation point of E.

5. Bonus: Let (x_n) be the sequence

   $$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}. \frac{1}{4}. \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \cdots$$

   Suppose $0 \leq a < b \leq 1$. Let $N(n, a, b)$ be the number of integers $j \leq n$ such that $x_j \in [a, b]$. Prove that

   $$\lim_{n\to\infty} \frac{N(n, a, b)}{n} = b - a.$$