Math 147 Assignment 3 - Due Friday October 8, 2010

1. (a) Suppose the sequence (x_n) converges to L and the sequence (y_n) converges to K. Prove that the sequence $(x_n + y_n)$ converges to $L + K$.

   (b) Suppose $a \leq z_n \leq b$ and $(z_n) \to s$. Prove $a \leq s \leq b$.

   (c) Suppose

$$w_n = \frac{(n-1)\cos(n^2+n+1)}{2n-1}.$$

   Does the sequence (w_n) have a convergent subsequence? Explain.

2. (a) Decide if the following sequences converge and if they do find the limit. Justify your answer.

   (i) $((-1)^n + \frac{1}{n}))_{n=1}^{\infty}$

   (ii) $(\frac{n!}{n^n})_{n=1}^{\infty}$

   (b) Suppose $x_n \to 0$ and $x_n \neq 0$ for any $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} x_n \sin\frac{1}{x_n} = 0$.

3. Decimal Expansion of a Real Number: (a) Let r be a real number with $0 \leq r \leq 1$. Let

$$r_1 = \max\{k \in \{0, 1, 2, ..., 9\} : k/10 \leq r\}$$

   and $s_1 = r_1/10$. Now proceed recursively. Assume we have defined $r_1, r_2, ..., r_n$ and put $s_n = \sum_{i=1}^{n} r_i/10^i$. Let

$$r_{n+1} = \max\{k \in \{0, 1, 2, ..., 9\} : k/10^{n+1} \leq r - s_n\}$$

   and put $s_{n+1} = \sum_{i=1}^{n+1} r_i/10^i$. Show that the sequence $(s_n)_{n=1}^{\infty}$ converges and that $\lim_{n\to\infty} s_n = r$.

   (This gives the decimal expansion of r and normally we would write $r = .r_1r_2r_3...$)

   (b) Conversely, suppose (a_n) is a sequence with each $a_n \in \{0, 1, 2, ..., 9\}$. Let $t_n = \sum_{i=1}^{n} a_i/10^i$. Show that the sequence (t_n) converges and if $t = \lim_{n\to\infty} t_n$ then $t \in [0, 1]$.

4. (a) Suppose that $a_n \geq 0$ and that $\lim_{n\to\infty} a_n = L$. Show that $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$.

   (Hint: Do the cases $L = 0$ and $L > 0$ separately. For $L > 0$ rationalize $\sqrt{a_n} - \sqrt{L}$)

   (b) Let $x_1 = 1$ and let $x_{n+1} = \sqrt{3 + 2x_n}$. Show that (x_n) converges and find the limit. (Hint: MCT)

5. Suppose $0 < a_1 < b_1$ and recursively define $a_{n+1} = \sqrt{a_n b_n}$ and $b_{n+1} = (a_n + b_n)/2$. Show that both sequences (a_n) and (b_n) converge and that they have the same limit.

6. **Bonus:** Prove that a sequence (x_n) converges to L if and only if every subsequence of (x_n) has a further subsequence that converges to L.