Math 147 Assignment 4 - **Due Friday October 15, 2010**

1. Using the definition prove : (a) $\lim_{x\to 1} \frac{x^3-3x+2}{x^2-1} = 0$

   (b) $\lim_{x\to 1} \frac{|x|-|x-2|}{x-1} = 2$

2. Evaluate : (a) $\lim_{x\to 0} \frac{\sqrt{x+p}-\sqrt{p}}{x}$ where $p > 0$

   (b) $\lim_{x\to 0} \frac{\sin 4x}{\tan 2x}$

3. Find the value(s) of c that will ensure the function f is continuous at $p = 0$ where

$$f(x) = \begin{cases} \frac{c}{2x-2} & \text{if } x \geq 0 \\ \frac{\sin|x|}{cx} & \text{if } x < 0 \end{cases}$$

4. A function g is said to be Lipschitz if there is some constant M such that

$$|g(x) - g(y)| \leq M\,|x - y| \text{ for all } x, y.$$

   Prove that a Lipschitz function is continuous.

5. Suppose

$$F(x) = \begin{cases} \frac{1}{2^n} & \text{if } x = \frac{b}{2^n} \text{ for an odd integer } b \\ 0 & \text{otherwise} \end{cases}.$$

   Show that F is discontinuous at p if $p = b/2^n$ for some odd integer b and is continuous at all other points p.

6. We say $\lim_{x\to\infty} f(x) = L$ (for $L \in \mathbb{R}$) if for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|f(x) - L| < \varepsilon$ whenever $x \geq N$. Use this definition to prove

$$\lim_{x\to\infty} \frac{x^2+3x-1}{6x^2-x+2} = \frac{1}{6}.$$

7. Suppose $a_1 = 1/2$ and $a_{n+1} = 1/(2 + a_n)$. Show the sequence (a_n) is Cauchy and find its limit. Hint: First prove $|a_{n+1} - a_{n+2}| < |a_n - a_{n+1}|/4$.

   Note: This limit is the so-called infinite continued fraction

$$\cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}$$

8. Bonus: Consider the polynomial $p(x) = x^3 + 5x - 1$. Let $0 < x_1 < 1$. Set

$$x_{n+1} = \frac{1}{5}(1 - x_n^3).$$

   Show that (x_n) converges and if $L = \lim_{n\to\infty} x_n$, then $p(L) = 0$.