Math 147 Assignment 6 - Due Friday October 29, 2010

1. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are increasing functions. Is $f \cdot g$ increasing? What about $f \circ g$?

2. The function $f : \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists $d \in \mathbb{R}$ with $f(x+d) = f(x)$ for all $x \in \mathbb{R}$. Suppose f is continuous and periodic. Prove that f attains maximum and minimum values.

3. Define f on $A = [0,1] \cup (2,3]$ by

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le 1 \\ x - 1 & \text{for } 2 < x \le 3 \end{cases} .$$

   (a) Show that f is continuous, $1-1$ and maps A onto $[0,2]$.

   (b) Show that f^{-1} is not continuous. (This shows the necessity of the domain of f being an interval in our theorem about the continuity of the inverse of a continuous function.)

4. For a function $f : [0, \infty) \to \mathbb{R}$ we say that $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that $|f(x) - L| < \varepsilon$ for all $x > N$.

   Suppose that f is continuous and $\lim_{x\to\infty} f(x) = f(0)$. Prove that f attains maximum and minimum values.

5. A set $U \subseteq \mathbb{R}$ is said to be *open* if whenever $x \in U$ there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq U$. A set F is said to be *closed* if its complement, $F^c = \{x \in \mathbb{R} : x \notin F\}$, is open.

   (a) Show that every interval of the form (a, b) is open (as defined above).

   (b) Show that if $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets, then

$$\bigcup_{\alpha \in I} U_\alpha = \{x \in \mathbb{R} : x \in U_\alpha \text{ for some } \alpha\}$$

   is also open.

   (c) Show that every interval of the form $[a, b]$ is closed.

   (d) Prove that a set $F \subseteq \mathbb{R}$ is closed if and only if whenever (x_n) is a sequence in F that converges to some x_0, then $x_0 \in F$.

   (e) A set $K \subseteq \mathbb{R}$ is called *compact* if every sequence (x_n) in K has a subsequence that converges to some $x_0 \in K$. Show that K is compact if and only if K is closed and bounded.