MATH 148 **Assignment 1** Due: Friday, January 21

1. Show that the function $f(x) = x$ is integrable over $[0, 1]$, and compute $\displaystyle\int_0^1 f$.

   I did such a problem in class for the function $g(x) = x^2$.

2. Let f on the interval $[0, 2]$ be defined by

$$f(x) = \begin{cases} 1 \text{ when } 0 \leq x < 1 \\ 0 \text{ when } 1 \leq x \leq 2 \end{cases} .$$

   Show from the definition of integrability that f is integrable and find its integral.

3. The following linearity property of integrals is very important and can be assumed for this problem. If f, g are integrable functions over $[a, b]$, then so is their sum $f + g$; and if c is any constant number, then the rescaled function cf is also integrable. Furthermore $\int_a^b (f + g) = \int_a^b f + \int_a^b g$ and $\int_a^b cf = c \int_a^b f$. We will discuss the proof in due course.

   Using this linearity and an example we did in class, prove that any integrable function f can be changed at a finite number of points in the interval $[a, b]$ without changing its integrability or its integral.

   Hint. It suffices to show the result when we change the integrable function f at only one point c in $[a, b]$. (Why?)

   Aside. The linearity statement above that sums and multiples of integrable functions are integrable is just the statement that the set of integrable functions over $[a, b]$ is a *vector space*.

4. Prove that the function $f(x) = \dfrac{\sin x}{1 + x^2}$ is uniformly continuous over $\mathbb{R}$.

5. Prove that the function $f(x) = \dfrac{1}{x}$ is not uniformly continuous over the open interval $(0, 1)$.

6. (a) Let $f(x) = \dfrac{1}{1 + x^2}$ over the interval $[0, 1]$. Explain very briefly why f is integrable.

If $\mathcal{P}_n$ is the uniform partition of $[0, 1]$ indicated below.

$$\mathcal{P}_n : 0 = \frac{0}{n} < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1$$

Find a value of n so that the upper sum $R(f, \mathcal{P}_n)$ estimates $\int_0^1 f$ with error at most $1/50$ and explain why your n will do the job.

(b) Repeat problem (a) for $g(x) = e^{-x^2}$ over $[0, 1]$.

7. (a) If f, g are integrable over $[a, b]$ and $f(x) \leq g(x)$ for all x in $[a, b]$, show that $\int_a^b f \leq \int_a^b g$.
In other words: "higher functions have higher integrals".

This monotonicity property of integrals should be kept in mind, because it is important.

Hint. First show the appropriate inequality for upper sums.

(b) Prove that $0 \leq \int_0^1 \sin(x^2)\, dx \leq \frac{1}{3}$.
Hint. Squeeze $\sin(x^2)$ between two functions whose integrals you know.

8. The next question involves a bit of logical thinking. It's not an inspiring question, but I hope it reinforces your abilities to think clearly.

Here's two ways of saying that a function f is uniformly continuous on an interval I.

A. For every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(p)| < \epsilon \text{ whenever } x, p \in I \text{ and } |x - p| < \delta.$$

B. For every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(p)| \leq \epsilon \text{ whenever } x, p \in I \text{ and } |x - p| \leq \delta.$$

Show that f satisfies property A if and only if f satisfies property B.

BONUSES.

Bonus problems should **NOT** be handed in with the regular assignment. Hand them in separately to me as soon as you get them done. For bonus problems, you should try to do them without getting help. Also be sure to get the regular assignments done well before you worry about the bonus problems.

1. Hand in directly to me anytime before January 31.
   You may have encountered the function f defined on the interval $[0, 1]$ by the rule

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1/n & \text{if } x \text{ is the rational number } m/n \text{ written in lowest terms.} \end{cases}$$

   If you sketch a bit of it for fractions with small denominators you should be able to see a "popcorn" pattern. This function is continuous at all irrational numbers but discontinuous at all rational numbers. Decide if f is integrable over $[0, 1]$, and justify your answer. If the function is integrable, what is its integral?

2. let f be the function defined onthe interval