MATH 148 **Assignment 2** Due: Friday, January 21

1. Suppose that f is continuous and strictly increasing over $[0, 1]$ and that $f(0) = 0$ while $f(1) = 1$. If g is the inverse function of f why is g integrable over $[0, 1]$? (A one-line explanation will do.) Without doing any calculations whatsoever, find $\int_0^1 f + \int_0^1 g$.

2. (a) We saw in the tutorial that on any interval, functions with a bounded derivative are uniformly continuous. This is a simple application of the mean value theorem.

   Is there a function on some interval with an unbounded derivative that is still uniformly continuous?

   (b) Is there a non-differentiable function that is still uniformly continuous?

3. Find the area in the first quadrant enclosed by the curves $y = \dfrac{2}{1+x^2}$ and $y = x$. This is a routine exercise using the fundamental theorem.

4. The functions $y = \sin x$ and $y = \cos x$ intersect each other infinitely often. Find the area of the region enclosed by these functions between any two adjacent intersections.

5. You may be aware that:

$$x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120},$$

   for all $x \geq 0$. If not, take it for granted anyway. Exploit this information along with the monotonic property of integration to prove that $\int_0^{1/2} \sin(x^2)\, dx \approx \dfrac{111}{2688}$ with error that is no more than $\dfrac{1}{2703360}$.
   Problem #7 of Assignment 1 was a baby version of this.

6. If f is integrable over $[a, b]$, show that $|f|$ is also integrable over $[a, b]$. Then show that

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

   This very important fact is known as the *triangle inequality for integration*.

Hint.

For the first part show that $U(|f|, \mathcal{P}) - L(|f|, \mathcal{P}) \leq U(f, \mathcal{P}) - L(f, \mathcal{P})$. You might have a hard time finding good words to explain this. Some people lament that they know something but can't seem to explain it. My reply would be "If you really know it, then you can explain it."

For the second part observe that $\pm f \leq |f|$, then use linearity of integration as well as the information in Problem #7 on Assignment #1.

7. If $a < c < b$ and f is bounded on $[a, b]$ but is discontinuous just at c, show that f is integrable.

   Hint. Given $\epsilon > 0$, you want a partition $\mathcal{P}$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$. Let B be an upper bound for $|f|$. Take $\delta > 0$ so small that $4B\delta < \epsilon$ and the numbers $c - \delta, c + \delta$ stay inside $[a, b]$. Now use the fact f is continuous on $[a, c - \delta]$ and $[c + \delta, b]$ to get suitable partitions of the outside intervals $[a, c - \delta]$ and $[c + \delta, b]$. Notice that the middle interval $[-c - \delta, c + \delta]$ does not amount to much, and write a suitable partition $\mathcal{P}$ of $[a, b]$. Once you play around you might have to do some "$\epsilon/3$ adjustments".

   For example, this problem shows that the crazy function f given by $f(x) = \sin(1/x)$ and $f(0) = 0$ is integrable on $[-1, 1]$. It also shows that all functions that have a jump discontinuity, i.e. functions where the left limit is not the same as the right limit at some spot c, are integrable.

   The result above can also be adapted to show that all functions that are discontinuous at only a finite number of places are integrable. It is also true that if a function is discontinuous at countably many places then the function is integrable, but now we are getting into hard territory.

8. This is a theoretical problem that gets used from time to time.

   (a) Suppose that f is continuous over $[a, b]$, that $f(x) \geq 0$ for all x in $[a, b]$, and that $\displaystyle\int_a^b f = 0$. Prove that $f(x) = 0$ for all x in $[a, b]$.

   In geometric parlance, this problem says that if a continuous non-negative function has zero area underneath it, it had better be the zero function. We have seen examples in class of non-zero, non-negative functions whose integral was 0, but those functions were not continuous.

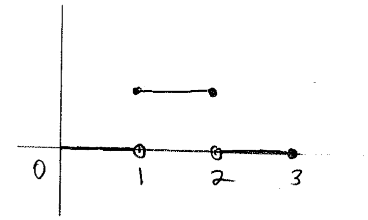
Hint. Here is the essential framework of the proof. Just fill in the details. Suppose f is not identically 0 so that $f(p) > 0$ for some p in $[a, b]$. Next explain how continuity gives a number $\delta > 0$ such that $f(x) > \frac{f(p)}{2}$ when $x \in (p-\delta, p+\delta)$. Then compare f with the function which takes the constant value $\frac{f(p)}{2}$ over the open interval $(p-\delta, p+\delta)$ and value 0 elsewhere in $[a, b]$.

(b) Does the result in part (a) hold if f is assumed to be merely integrable over $[a, b]$?

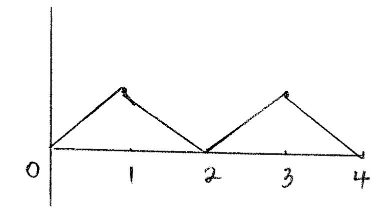
9. If f is integrable over $[a, b]$, then f is integrable over any subinterval $[a, x]$ where x runs between a and b. Take this for granted.

Thus we can speak of the integrals $\displaystyle\int_a^x f$. This function of x is called the *integral function*. This is a new and important way to make functions. For each function f as shown on the next page, make a sketch of its integral function over the indicated interval. Your sketches should indicate where the integral functions increase or decrease as well as the concavity behaviour of the integral functions. This is a purely descriptive exercise that requires neither calculations nor explanations.

(a)



(b)



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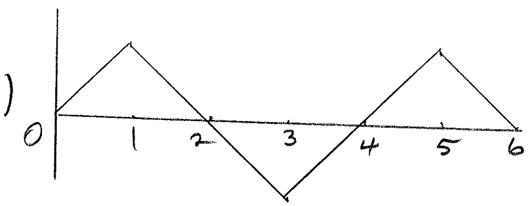

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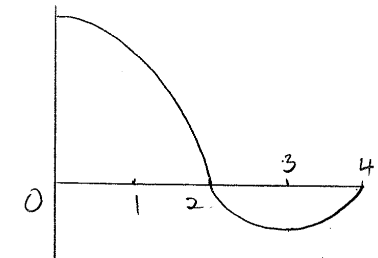
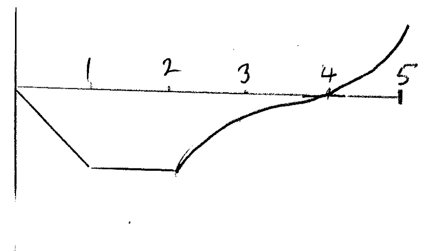


(e)



Keep in mind that $\int_a^x f$ is the "area under" f from a to x. Keep in mind that "area" needs to be subtracted when f becomes negative.

BONUSES

Bonus problems are not required to be handed in. They are meant to stimulate your interest. Bonus problems should not be handed in with the regular assignment. Hand them in separately to me as soon as you get them done. For bonus problems, you should try to do them without getting help. Also be sure to get the regular assignments done well before you worry about the bonus problems.

Hand these in before reading week.

2. Does there exist a continuous function on an interval I such that on every subinterval it is not monotonic?

3. If f, g are integrable over $[a, b]$, prove that $\left|\int_a^b fg\right| \leq \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}$.