MATH 148 **Assignment 4** Due: Friday, February 11

Reminder.

We have a $1\frac{1}{2}$ hour mid-term coming up:

Monday, February 14 from 7 to 8:30 p.m. in EIT 1015

It will count for 20% of your final grade. For the mid-term you should know everything we will have covered up to and including the second part of the Fundamental theorem and including this current assignment. Some of the longer and more tedious proofs that we did will not be on the exam. On UW-ACE I will post a short list of proofs that you should know.

In order to give you time to prepare for the mid-term and to give my TA's a break during Reading Week, there will be no assignment due on February 18. Also because of Reading Week, there will be no assignment due on February 25. After this assignment, the next assignment will be due on Friday, March 4. That one could be a bit longer than normal. The current assignment isn't that short either, so please start early to work on it. Despite the two-week gap in my assignments, make an effort to stay caught up with the material.

1. Find the derivative of these functions.

   (a) $g(x) = \displaystyle\int_3^{e^{x^2}} \ln(t^3)\, dt$ (b) $g(x) = \displaystyle\int_{\sin x}^{x^2} \sqrt{1+t^3}\, dt$

2. Find a function f and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\, dt = 2\sqrt{x} \text{ for all } x \geq 0.$$

3. (a) Prove that the function $f(x) = \displaystyle\int_{-1}^{x} \sqrt{1-t^2}\, dt$ is increasing on the interval $[-1, 1]$.
   Note. There is no need to do any calculations.

   (b) Sketch the graph of f over $[-1, 1]$.

   (c) What is the image of $[-1, 1]$ under the function f?

   (d) If g is the inverse function of f, find $g'(\pi/4)$.

4. If f is integrable over $[a, b]$, prove that the integral function $g(x) = \int_a^x f$ is continuous over $[a, b]$.
Hint. Don't forget that integrable functions are bounded by assumption.

Thus we see that integration improves the integrand.

- The integral function g of an integrable function f becomes a continuous function.
- By the Fundamental Theorem of Calculus-II, The integral function g of a continuous function f becomes a differentiable function.

Here come a number of exercises about so called **improper integrals**. *There is a fair bit of work to do in the upcoming problems.*

5. Suppose f is a function defined on the interval $[a, \infty)$ and that on each bounded subinterval $[a, x]$ the function is bounded and integrable. For example, this happens for sure when f is continuous. Thus we have the integral function $g(x) = \int_a^x f(t)\,dt$ defined for all $x \geq a$. The limit

$$\lim_{x \to \infty} \int_a^x f(t)\,dt$$

may or may not exist. When this limit exists we call this limit the *improper integral* of f over the interval $[a, \infty)$, and we write

$$\int_a^\infty f \quad \text{or} \quad \int_a^\infty f(t)\,dt \quad \text{to mean} \quad \lim_{x \to \infty} \int_a^x f(t)\,dt.$$

In this case we also say that the improper integral $\int_a^\infty f(t)\,dt$ *converges*. Presumably such integrals are called improper because of the unbounded interval $[a, \infty)$ over which we are trying to integrate our function.

(a) Does the improper integral $\int_1^\infty \frac{1}{1+t^2}\,dt$ converge? If so find its value.

(b) Does the improper integral $\int_e^\infty \frac{1}{t \ln t}\,dt$ converge?

(c) Compute $\displaystyle\int_e^\infty \frac{1}{t(\ln t)^2}\,dt$.

(d) Find the values of $p > 0$ for which the improper integrals $\displaystyle\int_1^\infty \frac{1}{t^p}\,dt$ converge.

6. Here comes a bit of theory about improper integrals.

   Suppose $f \geq 0$ on $[a, \infty)$ and let $g(x) = \displaystyle\int_a^x f$ be the integral function of f, which is also defined on $[a, \infty)$.

   (a) Show that g is increasing on $[a, \infty)$. (It need not be strictly).
   Hint. Use the definition of increasing functions, don't forget the splicing property, and remember that the integral of a non-negative function is a non-negative number.

   (b) If $f \geq 0$, show that $\displaystyle\int_a^\infty f = \lim_{x\to\infty} g(x)$ exists if and only if g is bounded on $[a, \infty)$.
   Hint. This problem applies to any increasing function and makes no use of the fact our g is an integral function. The proof is a total imitation of the proof that increasing bounded sequences converge. Just to get you started, suppose g is bounded, let L be the least upper bound of g, and note that for any $\epsilon > 0$ the number $L - \epsilon$ is no longer an upper bound of g...

   (c) Suppose $0 \leq f \leq h$ on $[a, \infty)$ and that $\displaystyle\int_a^\infty h$ converges.
   Prove that $\displaystyle\int_a^\infty f$ converges.
   This result is called the *comparison test* for the convergence of improper integrals. Just use part (b) above.

   (d) Make a comparison to prove that $\displaystyle\int_1^\infty \frac{1}{1+t^4}\,dt$ converges.

   (e) Prove that $\displaystyle\int_1^\infty \frac{5+\cos t}{\sqrt{t}}\,dt$ diverges.
   Hint. The comparison test along with problem 5(d) should help.

7. Here is another batch of problems on improper integrals.

   Again we have f defined on $[a, \infty)$ but we do not assume $f \geq 0$ on $[a, \infty)$.

(a) Use the obvious inequalities $0 \leq f + |f| \leq 2|f|$, along with preceding facts, to prove that if $\int_a^\infty |f|$ converges, so does $\int_a^\infty f$ converge.

(b) Use item (a) to prove that $\int_1^\infty \frac{\sin t}{t^{3/2}}\, dt$ converges.

(c) Prove that $\int_1^\infty \frac{\sin t}{\sqrt{t}}\, dt$ converges.

Hint. Item (a) won't help much but something extra sneaky, such as integration by parts, might.

BONUSES

Bonus problems are not required to be handed in. They are meant to stimulate your interest. If you do a bonus problem, hand it in separately to me as soon as you get it done. For bonus problems, you should try to do them without getting help. Also be sure to do a good job on the regular assignments before you worry about the bonus problems. I will try to mark the bonuses as soon as I can.

Hand this one in by the end of March.

5. Show that $\int_0^\infty \frac{\sin t}{t}\, dt$ converges to $\frac{\pi}{2}$.

   This is an interesting, hard and quite long problem. It appears on page 392 in the third edition of Spivak's book as problem #42, with several hints offered.