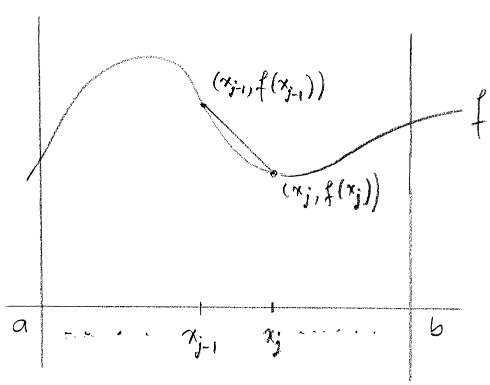
MATH 148 **Assignment 5** Due: Friday, March 4

1. This exercise is about **arc-lengths**. Please read the discussion below before doing the problems.

   Let f be a differentiable function defined on $[a, b]$ and such that the derivative f' is also continuous. These kinds of functions are usually called C^1-functions. We are interested in a formula for the arc-length formed by the graph of f. To that end partition the interval $[a, b]$ in the usual style:

   $$\mathcal{P} : a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

   For each $j = 1, \ldots, n$, the straight line segment running from the point $(x_{j-1}, f(x_{j-1}))$ to the point $(x_j, f(x_j))$ has length that approximates the arc-length of the graph of f over the interval $[x_{j-1}, x_j]$.



The length of this line segment is

$$\sqrt{(x_j - x_{j-1})^2 + (f(x_j) - f(x_{j-1}))^2} = \sqrt{1 + \left(\frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}}\right)^2}(x_j - x_{j-1}).$$

By the mean-value theorem applied to f over each interval $[x_{j-1}, x_j]$, this last quantity becomes

$$\sqrt{1 + f'(t_j)^2}(x_j - x_{j-1}) \text{ for some } t_j \text{ between } x_{j-1} \text{ and } x_j.$$

Hence we see that the Riemann sums $R\left(\sqrt{1 + (f')^2}, \mathcal{P}, t_1, \ldots, t_n\right)$ approximate our desired arc-length. Note that $\sqrt{1 + (f')^2}$ is continuous, since we assumed that f' is continuous. When all $x_j - x_{j-1} \to 0$, the resulting integral

$$\int_a^b \sqrt{1 + (f')^2}$$

will compute the desired arc-length. Often these integrals are brutal to calculate using the Fundamental Theorem of Calculus Part I, but here come a couple you can do.

(a) When a chain hangs loosely from two points of equal height, it forms a curve called a *catenary*. For example, a loose necklace. The word catenary is not surprising since it comes from the Latin "catena" meaning chain, but I digress. The famous arch in St. Louis also has the shape of a catenary. The shape of a catenary is modelled by the even function

$$y = \frac{e^x + e^{-x}}{2},$$

also known as the hyperbolic cosine.

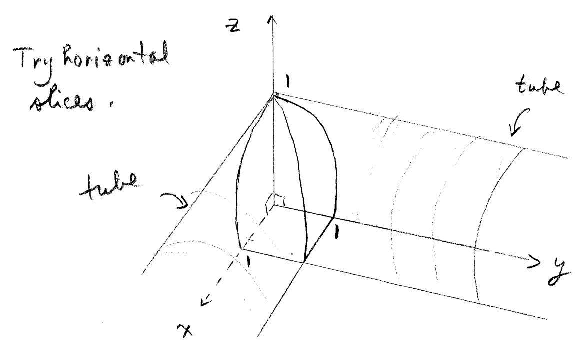
Sketch this catenary and then find its arc-length over the interval $[-1, 1]$.

(b) Find the arc-length of $y = e^x$ over the interval $[0, 1]$.

2. Two circular tubes of radius 1 intersect each other at right angles, with their central axes also crossing. In other words they meet in a fully symmetrical way.

Find the volume of the resulting solid of intersection.

Hint. The difficulty, if any, stems from a need for some 3-D imagination. Here is a picture of the portion of the solid that lies in the first octant in $\mathbb{R}^3$. I hope it helps.



3. Here we exploit the monotonicity property of integrals to discover some neat stuff. Since the principle applies to all integrals it applies to integral functions. Namely:

   If $f \le g$ on $[a, b)$ where $a < b \le \infty$, then $\displaystyle\int_a^x f \le \int_a^x g$ for all x in $[a, b)$.

   We shall refer the above principle as *lifting the inequality* $f \le g$ on the interval $[a, b)$

   (a) Lift the inequality $e^{-x} \le 1$ on $[0, \infty)$ a few times to show that

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} \le e^{-x} \le 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}.$$

for all $x \geq 0$.

(b) Use the information from part (a) to find a fraction that estimates $\int_0^1 e^{-t^2}\,dt$ with error at most $\dfrac{1}{216}$.

4. If a function f is uniformly continuous on a bounded, open interval (a, b), show that f is bounded over (a, b). Does this hold if f is merely continuous over (a, b)?
Hint. Take a $\delta > 0$ that works for $\epsilon = 1$, and then pick a finite number of points $p_1, \ldots, p_n$ in (a, b) such that every x in (a, b) is within δ of some p_j.

5. For any positive integers m and n, the functions $(\cos m!\pi x)^{2n}$ are certainly continuous. By way of contrast find the function you get when you calculate

$$\lim_{m\to\infty} \lim_{n\to\infty} (\cos m!\pi x)^{2n} \text{ for every } x.$$

Hint. Handle the case where x is irrational separately from the rational case.

The purpose of this exercise was to alert you to the fact that limits of continuous functions need not be continuous functions.

6. Suppose that $x_1, x_2, \ldots, x_n, \ldots$ is a decreasing sequence of non-negative numbers, and that $\sum_{n=1}^{\infty} x_n$ converges. Prove that $nx_n \to 0$.

Hint. Convergent sequences are Cauchy sequences, and vice versa.

7. Find an n such that $\displaystyle\sum_{k=n+1}^{\infty} \frac{1}{k^5} < 10^{-6}$.

8. Show that $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k \ln k}$ diverges. That's not hard to do with the integral test, but we want to get an idea of how s-l-o-w-l-y this series blows up.
If $s_n = \displaystyle\sum_{k=3}^{n} \frac{1}{k \ln k} \geq 10$, prove that $n \geq e^{e^9}$.

Hint. Compare the finite sum to an integral.

Do you think a computer can carry out the required number of additions to have s_n exceed 10 within an hour? Just give a brief opinion.

9. The famous number e is defined to be the sum of the series $\sum_{k=0}^{\infty} \frac{1}{k!}$. We may well know that $e \approx 2.718281828$. Here we ask ourselves how such estimates for e can be obtained. Let $s_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$.

   (a) Clearly

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots \\ &= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots\right) \\ &\leq \frac{1}{(n+1)!}\left(1 + \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots\right) \end{aligned}$$

   Use geometric series to complete the argument started above and show that $e - s_n \leq \frac{1}{n!n}$

   The above estimate for e using s_n is memorable stuff.

   (b) Show that s_{10} estimates e with error at most $\dfrac{1}{36288000}$.

   (c) Use the result of part (a) to prove that e is irrational.

   Hint. Suppose e is a rational number $\frac{p}{q}$. Then apply the inequality of part (a) for a suitably large n to get a contradiction. This very interesting item is just a bit harder.

BONUSES

You can hand these in these optional problems by the end of March.

6. On the interval $[-1, 1]$ define a sequence of polynomials recursively by:

$$f_0(x) = 0, \quad f_{n+1}(x) = f_n(x) + \frac{x^2 - f_n^2(x)}{2}$$

   (a) Find $f_n(x)$ for $n = 1, 2, 3$, and sketch these polynomials on graph together with $y = |x|$. Using MAPLE would be nice.

   (b) Show that for all n

$$|x| - f_{n+1}(x) = (|x| - f_n(x))\left(1 - \frac{|x| + f_n(x)}{2}\right)$$

(c) Prove that

$$0 \leq f_n(x) \leq f_{n+1}(x) \leq |x| \text{ for all } n.$$

Hint. Assume all three inequalities for $n-1$ and show all three for n. Of course you have to check the startup case where $n = 0$.

(d) Show that

$$|x| - f_n(x) \leq |x| \left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1} \text{ for all } n$$

(e) Show that the polynomials f_n converge uniformly on $[-1, 1]$ to the absolute value function.

To be able to tuck a polynomial uniformly close to a tight corner like this is quite a decent trick.

You should note that these polynomials that tend uniformly to $|x|$ on $[-1, 1]$ are not Taylor polynomials for $y = |x|$. After all the function $y = |x|$ does not even have a derivative at $x = 0$.

7. If $a_k > 0$ and $\sum_{k=1}^{\infty} a_k$ diverges and s_n is the n'th partial sum, show that $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ still diverges.